

# KÄHLER DIFFERENTIAL ALGEBRAS FOR 0-DIMENSIONAL SCHEMES

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**ABSTRACT.** Given a 0-dimensional scheme in a projective  $n$ -space  $\mathbb{P}^n$  over a field  $K$ , we study the Kähler differential algebra  $\Omega_{R_{\mathbb{X}}/K}$  of its homogeneous coordinate ring  $R_{\mathbb{X}}$ . Using explicit presentations of the modules  $\Omega_{R_{\mathbb{X}}/K}^m$  of Kähler differential  $m$ -forms, we determine many values of their Hilbert functions explicitly and bound their Hilbert polynomials and regularity indices. Detailed results are obtained for subschemes of  $\mathbb{P}^1$ , fat point schemes, and subschemes of  $\mathbb{P}^2$  supported on a conic.

## 1. INTRODUCTION

In the paper [DK], G. de Dominicis and the first author introduced the application of Kähler differential modules to the study of 0-dimensional subschemes  $\mathbb{X}$  of a projective space  $\mathbb{P}^n$  over a field  $K$  of characteristic zero. They showed that this graded module over the homogeneous coordinate ring  $R_{\mathbb{X}}$  contains numerical and algebraic information which is not readily available from the homogeneous vanishing ideal or from  $R_{\mathbb{X}}$ . Later, in [KLL], the authors extended and refined these techniques for fat point schemes in  $\mathbb{P}^n$ . Following the classical construction described by E. Kunz in his book [Kun], it is natural to define the Kähler differential algebra  $\Omega_{R_{\mathbb{X}}/K} = \bigoplus_{m \in \mathbb{N}} \Omega_{R_{\mathbb{X}}/K}^m$  of  $\mathbb{X}$  as the exterior algebra of its Kähler differential module  $\Omega_{R_{\mathbb{X}}/K}^1$ . This invites the question whether the Kähler differential algebra contains numerical and algebraic information about  $\mathbb{X}$  which is not readily available in  $R_{\mathbb{X}}$  or in  $\Omega_{R_{\mathbb{X}}/K}^1$ . Thus the following example provided the initial spark to ignite the curiosity of the authors.

**Example 1.1** (See Example 2.6). Let  $\mathbb{X}$  and  $\mathbb{Y}$  be two sets of six reduced  $K$ -rational points in  $\mathbb{P}^2$  such that  $\mathbb{X}$  is contained in a non-singular conic, and such that  $\mathbb{Y}$  consists of three points on a line and three points on another line. Then the Hilbert functions of  $R_{\mathbb{X}}$  and  $R_{\mathbb{Y}}$  agree, as do the Hilbert functions of  $\Omega_{R_{\mathbb{X}}/K}^1$  and  $\Omega_{R_{\mathbb{Y}}/K}^1$ . However, the Hilbert functions of  $\Omega_{R_{\mathbb{X}}/K}^2$  and  $\Omega_{R_{\mathbb{Y}}/K}^2$  are different, and also the Hilbert functions of  $\Omega_{R_{\mathbb{X}}/K}^3$  and  $\Omega_{R_{\mathbb{Y}}/K}^3$  disagree:

$$\begin{aligned} \text{HF}_{\Omega_{R_{\mathbb{X}}/K}^2} &: 0 \ 0 \ 3 \ 6 \ 4 \ 1 \ 0 \ 0 \cdots, & \text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^2} &: 0 \ 0 \ 3 \ 6 \ 5 \ 1 \ 0 \ 0 \cdots, \\ \text{HF}_{\Omega_{R_{\mathbb{X}}/K}^3} &: 0 \ 0 \ 0 \ 1 \ 0 \ 0 \cdots, & \text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^3} &: 0 \ 0 \ 0 \ 1 \ 1 \ 0 \cdots. \end{aligned}$$

So, the Hilbert functions of the exterior powers of  $\Omega_{R_{\mathbb{X}}/K}^1$  “know” whether  $\mathbb{X}$  is contained in an irreducible or a reducible conic.

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This observation motivated the studies underlying this paper. Let us now outline its contents in more detail. In the second section we start by recalling the definitions of the Kähler differential module  $\Omega_{R_{\mathbb{X}}/K}^1$  and the Kähler differential algebra  $\Omega_{R_{\mathbb{X}}/K} = \bigwedge_{R_{\mathbb{X}}}(\Omega_{R_{\mathbb{X}}/K}^1)$  of a 0-dimensional subscheme  $\mathbb{X}$  of  $\mathbb{P}^n$ . As explained in [Kun], we can calculate an explicit presentation of  $\Omega_{R_{\mathbb{X}}/K}^m = \bigwedge_{R_{\mathbb{X}}}^m(\Omega_{R_{\mathbb{X}}/K}^1)$  for every  $m \geq 1$ . Moreover, we show that  $\Omega_{R_{\mathbb{X}}/K}^m = \langle 0 \rangle$  for  $m > n + 1$ , provide a simplified presentation for  $\Omega_{R_{\mathbb{X}}/K}^{n+1}$ , and show that the Koszul complex yields an exact sequence

$$0 \longrightarrow \Omega_{R_{\mathbb{X}}/K}^{n+1} \longrightarrow \Omega_{R_{\mathbb{X}}/K}^n \longrightarrow \cdots \longrightarrow \Omega_{R_{\mathbb{X}}/K}^2 \longrightarrow \Omega_{R_{\mathbb{X}}/K}^1 \longrightarrow \mathfrak{m}_{\mathbb{X}} \longrightarrow 0$$

where  $\mathfrak{m}_{\mathbb{X}}$  is the homogeneous maximal ideal  $\mathfrak{m}_{\mathbb{X}} = \langle x_0, \dots, x_n \rangle$  of  $R_{\mathbb{X}}$ .

In Section 3 we have a brief glance at the case  $n = 1$ , i.e., at 0-dimensional subschemes of a projective line. Unsurprisingly, in this case the Hilbert functions and regularity indices of  $\Omega_{R_{\mathbb{X}}/K}^1$  and  $\Omega_{R_{\mathbb{X}}/K}^2$  can be written down explicitly.

In Section 4 we look at the Hilbert function of  $\Omega_{R_{\mathbb{X}}/K}^m$  in special degrees. We provide explicit values in low degrees, show that the Hilbert polynomial (i.e., the value in high degrees) is constant, and examine monotonicity in intermediate degrees. These insights are accompanied by a bound for the regularity index of  $\Omega_{R_{\mathbb{X}}/K}^m$  in terms of the regularity index of  $\Omega_{R_{\mathbb{X}}/K}^1$  in Section 5.

Then, in the next four sections, we look at the modules of Kähler differential  $m$ -forms for fat point schemes  $\mathbb{W}$ . Such schemes are defined by ideals of the form  $I_{\mathbb{W}} = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s}$ , where the ideals  $\wp_i$  are the vanishing ideals of distinct  $K$ -rational points in  $\mathbb{P}^n$ . In Section 6 we prove a regularity bound for  $\Omega_{R_{\mathbb{W}}/K}^m$  which uses the regularity index of the fattening of  $\mathbb{W}$ , i.e., the scheme  $\mathbb{V}$  defined by  $I_{\mathbb{V}} = \wp_1^{m_1+1} \cap \cdots \cap \wp_s^{m_s+1}$ . For fat point schemes, we also give bounds on some specific values of the Hilbert polynomial of  $\Omega_{R_{\mathbb{W}}/K}^m$ . In the reduced case (i.e., when  $m_1 = \cdots = m_s = 1$ ), these values are zero for  $m \geq 2$ , but as soon as one of the exponents  $m_i$  satisfies  $m_i \geq 2$ , not all of these values are zero anymore. Thus the property of  $\mathbb{W}$  to be reduced is reflected in the values of the Hilbert polynomials of  $\Omega_{R_{\mathbb{W}}/K}^m$  (see Cor. 7.5). More generally, Prop. 7.4 provides upper and lower bounds for these Hilbert polynomials.

For the highest non-zero module of Kähler differentials  $\Omega_{R_{\mathbb{W}}/K}^{n+1}$ , we can sometimes determine its Hilbert polynomial explicitly. More precisely, we have formulas for schemes  $\mathbb{W}$  contained in a hyperplane (see Prop. 8.1) and for equimultiple schemes  $\mathbb{W}$  (see Thm. 8.3). Another case in which we have more detailed information is the module  $\Omega_{R_{\mathbb{W}}/K}^2$  for an equimultiple fat point scheme  $\mathbb{W}$  (i.e., a scheme satisfying  $m_1 = \cdots = m_s$ ). In this case we can extract the value of the Hilbert polynomial of  $\Omega_{R_{\mathbb{W}}/K}^2$  from a complex connecting it to its fattening and second fattening (see Prop. 8.5). We end the discussion of Hilbert polynomials with a conjecture for their value for  $\Omega_{R_{\mathbb{W}}/K}^{n+1}$ .

In Section 9, a rich and detailed set of results describes the Hilbert functions of  $\Omega_{R_{\mathbb{W}}/K}^m$ , where  $m = 1, 2, 3$ , in the case of a fat point scheme  $\mathbb{W}$  in  $\mathbb{P}^2$  supported on a non-singular conic. In this case, the Hilbert function of  $\Omega_{R_{\mathbb{W}}/K}^1$  can be computed explicitly from the Hilbert functions of suitable fat point schemes (see Thm. 9.2). If  $\mathbb{W}$  is an equimultiple fat point scheme, we construct a special homogeneous system of generators of  $I_{\mathbb{W}}$  in Prop. 9.4 and use it to compute the Hilbert function of  $\Omega_{R_{\mathbb{W}}/K}^3$  explicitly (see Thm. 9.6 and Prop. 9.7). Consequently, we can use the

exact sequence given by the Koszul complex and determine the Hilbert function of  $\Omega_{R_{\mathbb{W}}/K}^2$  explicitly (see Prop. 9.9).

Finally, in the last section we point out the relation between the Kähler differential algebra  $\Omega_{R_{\mathbb{X}}/K}$  and the relative Kähler differential algebra  $\Omega_{R_{\mathbb{X}}/K[x_0]}$  and use it to deduce many properties of the Hilbert function, the Hilbert polynomial, and the regularity index of  $\Omega_{R_{\mathbb{X}}/K[x_0]}$  (see Propositions 10.1, 10.2 and 10.3).

Throughout the paper we illustrate all results with explicitly computed examples. The necessary calculations were performed using the second author's package for the computer algebra system ApCoCoA (see [ApC]). Unless explicitly stated otherwise, we adhere to the definitions and notation introduced in [KR1, KR2] and [Kun].

## 2. DEFINITION AND BASIC PROPERTIES

Throughout this paper we work over a field  $K$  of characteristic zero. By  $\mathbb{P}^n$  we denote the projective  $n$ -space over  $K$ . The homogeneous coordinate ring of  $\mathbb{P}^n$  is  $S = K[X_0, \dots, X_n]$ . It is equipped with the standard grading  $\deg(X_i) = 1$  for  $i = 0, \dots, n$ . Let  $\mathbb{X}$  be a 0-dimensional scheme in  $\mathbb{P}^n$ , and let  $I_{\mathbb{X}}$  be the (saturated) homogeneous vanishing ideal of  $\mathbb{X}$ . Then the homogeneous coordinate ring of  $\mathbb{X}$  is  $R_{\mathbb{X}} = S/I_{\mathbb{X}}$ . The ring  $R_{\mathbb{X}} = \bigoplus_{i \geq 0} (R_{\mathbb{X}})_i$  is a standard graded  $K$ -algebra. Its enveloping algebra is  $R_{\mathbb{X}} \otimes_K R_{\mathbb{X}} = \bigoplus_{i \geq 0} (\bigoplus_{j+k=i} (R_{\mathbb{X}})_j \otimes (R_{\mathbb{X}})_k)$ . By  $\mathcal{J}$  we denote the kernel of the homogeneous  $R_{\mathbb{X}}$ -linear map of degree zero  $\mu : R_{\mathbb{X}} \otimes_K R_{\mathbb{X}} \rightarrow R_{\mathbb{X}}$  given by  $\mu(f \otimes g) = fg$ . It is well known that  $\mathcal{J}$  is the homogeneous ideal of  $R_{\mathbb{X}} \otimes_K R_{\mathbb{X}}$  generated by  $\{x_i \otimes 1 - 1 \otimes x_i \mid 0 \leq i \leq n\}$ , where  $x_i$  is the image of  $X_i$  in  $R_{\mathbb{X}}$  for  $i = 0, \dots, n$ . In this paper we are interested in looking at the algebraic structure and Hilbert function of the following objects.

- Definition 2.1.** (a) The graded  $R_{\mathbb{X}}$ -module  $\Omega_{R_{\mathbb{X}}/K}^1 = \mathcal{J}/\mathcal{J}^2$  is called the **module of Kähler differential 1-forms** of  $R_{\mathbb{X}}/K$ , or simply the **module of Kähler differentials**.
- (b) The homogeneous  $K$ -linear map  $d : R_{\mathbb{X}} \rightarrow \Omega_{R_{\mathbb{X}}/K}^1$  given by  $f \mapsto f \otimes 1 - 1 \otimes f + \mathcal{J}^2$  is called the **universal derivation** of  $R_{\mathbb{X}}/K$ .
- (c) The  $m$ -th exterior power of  $\Omega_{R_{\mathbb{X}}/K}^1$  over  $R_{\mathbb{X}}$  is called the **module of Kähler differential  $m$ -forms** of  $R_{\mathbb{X}}/K$  and is denoted by  $\Omega_{R_{\mathbb{X}}/K}^m$ .
- (d) The direct sum  $\Omega_{R_{\mathbb{X}}/K} := \bigoplus_{m \in \mathbb{N}} \Omega_{R_{\mathbb{X}}/K}^m$  is an  $R_{\mathbb{X}}$ -algebra. It is called the **Kähler differential algebra** of  $R_{\mathbb{X}}/K$ . Here we use  $\Omega_{R_{\mathbb{X}}/K}^0 = R_{\mathbb{X}}$ .

More generally, for any graded  $K$ -algebra  $T/S$ , we can define the module of Kähler differential  $m$ -forms  $\Omega_{T/S}^m$  and the Kähler differential algebra  $\Omega_{T/S}$  in analogously (cf. [Kun, Section 2]). The Kähler differential algebra of  $\Omega_{R_{\mathbb{X}}/K}$  is in fact a bigraded  $K$ -algebra whose homogeneous component in degree  $(m, d)$  is given by  $(\Omega_{R_{\mathbb{X}}/K}^m)_d$ . Notice that we have  $\deg(dx_i) = \deg(x_i) = 1$  for  $i = 0, \dots, n$ . For  $m \geq 0$ , the graded  $R_{\mathbb{X}}$ -module  $\Omega_{R_{\mathbb{X}}/K}^m$  is finitely generated and its Hilbert function is defined by

$$\text{HF}_{\Omega_{R_{\mathbb{X}}/K}^m}(i) = \dim_K(\Omega_{R_{\mathbb{X}}/K}^m)_i \quad \text{for all } i \in \mathbb{Z}.$$

Note that  $\Omega_{R_{\mathbb{X}}/K}^0 = R_{\mathbb{X}}$  and  $\Omega_{R_{\mathbb{X}}/K}^1 = R_{\mathbb{X}}dx_0 + \dots + R_{\mathbb{X}}dx_n$ . Hence we obtain  $\Omega_{R_{\mathbb{X}}/K}^m = \langle 0 \rangle$  for  $m > n + 1$  and  $\Omega_{R_{\mathbb{X}}/K} = \bigoplus_{m=0}^{n+1} \Omega_{R_{\mathbb{X}}/K}^m$ . Furthermore, there

is a presentation of  $\Omega_{R_{\mathbb{X}}/K}$  as  $\Omega_{R_{\mathbb{X}}/K} \cong \Omega_{S/K}/\langle I_{\mathbb{X}}, dI_{\mathbb{X}} \rangle \Omega_{S/K}$  (cf. [Kun, Proposition 4.12]). From this we deduce the following presentation of the module of Kähler differential  $m$ -forms.

**Proposition 2.2.** *Let  $m \geq 1$  and let  $\{F_1, \dots, F_r\}$  be a homogeneous system of generators of  $I_{\mathbb{X}}$ . The graded  $R_{\mathbb{X}}$ -module  $\Omega_{R_{\mathbb{X}}/K}^m$  has a presentation*

$$\Omega_{R_{\mathbb{X}}/K}^m \cong \Omega_{S/K}^m / (I_{\mathbb{X}} \Omega_{S/K}^m + dI_{\mathbb{X}} \Omega_{S/K}^{m-1})$$

where  $I_{\mathbb{X}} \Omega_{S/K}^m + dI_{\mathbb{X}} \Omega_{S/K}^{m-1}$  is generated by

$$\begin{aligned} & \{F_j dX_{i_1} \wedge \dots \wedge dX_{i_m} \mid 1 \leq j \leq r, 0 \leq i_1 < \dots < i_m \leq n\} \\ & \cup \{dF_j \wedge dX_{j_1} \wedge \dots \wedge dX_{j_{m-1}} \mid 1 \leq j \leq r, 0 \leq j_1 < \dots < j_{m-1} \leq n\}. \end{aligned}$$

In the case  $m = n + 1$ , the presentation of  $\Omega_{R_{\mathbb{X}}/K}^{n+1}$  can be described explicitly as follows.

**Corollary 2.3.** *Let  $\{F_1, \dots, F_r\}$  be a homogeneous system of generators of  $I_{\mathbb{X}}$ . There is an isomorphism of graded  $R_{\mathbb{X}}$ -modules*

$$\Omega_{R_{\mathbb{X}}/K}^{n+1} \cong (S / \langle \frac{\partial F_j}{\partial X_i} \mid 0 \leq i \leq n, 1 \leq j \leq r \rangle) (-n - 1).$$

*Proof.* Note that  $\Omega_{S/K}^{n+1}$  is a free  $S$ -module of rank 1 with basis  $\{dX_0 \wedge \dots \wedge dX_n\}$ , and so  $\Omega_{S/K}^{n+1} \cong S(-n-1)$ . For  $F \in I_{\mathbb{X}}$  and  $G \in S$ , we have  $FdG = d(FG) - GdF \in dI_{\mathbb{X}}$ . It follows that  $I_{\mathbb{X}} \Omega_{S/K}^m \subseteq dI_{\mathbb{X}} \Omega_{S/K}^{m-1}$  for all  $m \geq 1$ . Let  $I = \langle \frac{\partial F_j}{\partial X_i} \mid 0 \leq i \leq n, 1 \leq j \leq r \rangle$ . We need to show that  $dI_{\mathbb{X}} \Omega_{S/K}^n = IdX_0 \wedge \dots \wedge dX_n$ . Clearly, we have

$$\frac{\partial F_j}{\partial X_i} dX_0 \wedge \dots \wedge dX_n = (-1)^i dF_j \wedge dX_0 \wedge \dots \wedge \widehat{dX_i} \wedge \dots \wedge dX_n \in dI_{\mathbb{X}} \Omega_{S/K}^n$$

where  $\widehat{dX_i}$  indicates that  $dX_i$  is omitted in the wedge product. Hence we get the inclusion  $dI_{\mathbb{X}} \Omega_{S/K}^n \supseteq IdX_0 \wedge \dots \wedge dX_n$ . For the other inclusion, let  $F \in I_{\mathbb{X}}$ , and let  $\{i_1, \dots, i_n\} \subseteq \{0, \dots, n\}$ . Write  $F = G_1 F_1 + \dots + G_r F_r$  with  $G_1, \dots, G_r \in S$ . Then we have

$$\begin{aligned} dF - (F_1 dG_1 + \dots + F_r dG_r) &= G_1 dF_1 + \dots + G_r dF_r \\ &= G_1 \sum_{i=0}^n \frac{\partial F_1}{\partial X_i} dX_i + \dots + G_r \sum_{i=0}^n \frac{\partial F_r}{\partial X_i} dX_i \end{aligned}$$

and hence  $(dF - (F_1 dG_1 + \dots + F_r dG_r)) \wedge dX_{i_1} \wedge \dots \wedge dX_{i_n} \in IdX_0 \wedge \dots \wedge dX_n$ . Since the field  $K$  has characteristic zero, for  $j = 1, \dots, r$ , Euler's relation yields that  $F_j = \frac{1}{\deg(F_j)} \sum_{i=0}^n X_i \frac{\partial F_j}{\partial X_i} \in I$ . In particular, we have  $I_{\mathbb{X}} \subseteq I$ . This implies  $dF \wedge dX_{i_1} \wedge \dots \wedge dX_{i_n} \in IdX_0 \wedge \dots \wedge dX_n$ . Therefore we get the equality  $dI_{\mathbb{X}} \Omega_{S/K}^n = IdX_0 \wedge \dots \wedge dX_n$ , and the claim follows readily.  $\square$

Now let  $\mathfrak{m}_{\mathbb{X}} = \langle x_0, \dots, x_n \rangle$  be the homogeneous maximal ideal of  $R_{\mathbb{X}}$ , and let  $e : R_{\mathbb{X}} \rightarrow R_{\mathbb{X}}$  be the **Euler derivation** of  $R_{\mathbb{X}}/K$  given by  $f \mapsto i \cdot f$  for  $f \in (R_{\mathbb{X}})_i$ . By universal property of  $\Omega_{R_{\mathbb{X}}/K}^1$  (cf. [Kun, Section 1]), there is a unique homogeneous  $R_{\mathbb{X}}$ -linear map  $\gamma : \Omega_{R_{\mathbb{X}}/K}^1 \rightarrow R_{\mathbb{X}}$  such that  $e = \gamma \circ d$ . In particular, we have  $\gamma(dx_i) = x_i$  for all  $i = 0, \dots, n$  and  $\gamma(df) = \deg(f) \cdot f$  for every homogeneous element  $f \in R_{\mathbb{X}} \setminus \{0\}$ . The Koszul complex of  $\gamma$  is the complex

$$\dots \xrightarrow{\gamma} \Omega_{R_{\mathbb{X}}/K}^2 \xrightarrow{\gamma} \Omega_{R_{\mathbb{X}}/K}^1 \xrightarrow{\gamma} \mathfrak{m}_{\mathbb{X}} \longrightarrow 0$$

where  $\gamma : \Omega_{R_{\mathbb{X}}/K}^m \rightarrow \Omega_{R_{\mathbb{X}}/K}^{m-1}$  is a homogeneous  $R_{\mathbb{X}}$ -linear map defined by

$$\gamma(\omega_1 \wedge \cdots \wedge \omega_m) = \sum_{j=1}^m (-1)^{j+1} \gamma(\omega_j) \cdot \omega_1 \wedge \cdots \wedge \widehat{\omega_j} \wedge \cdots \wedge \omega_m$$

for all  $\omega_1, \dots, \omega_m \in \Omega_{R_{\mathbb{X}}/K}^1$ , and where  $\gamma(\omega \wedge \omega') = \gamma(\omega) \wedge \omega' + (-1)^m \omega \wedge \gamma(\omega')$  for  $\omega \in \Omega_{R_{\mathbb{X}}/K}^m$  and  $\omega' \in \Omega_{R_{\mathbb{X}}/K}^k$  (cf. [BH, 1.6.1-2]). In our setting, this complex is an exact sequence, as the following proposition shows.

**Proposition 2.4.** *The Koszul complex*

$$(\mathcal{K}) \quad 0 \longrightarrow \Omega_{R_{\mathbb{X}}/K}^{n+1} \xrightarrow{\gamma} \Omega_{R_{\mathbb{X}}/K}^n \longrightarrow \cdots \longrightarrow \Omega_{R_{\mathbb{X}}/K}^2 \xrightarrow{\gamma} \Omega_{R_{\mathbb{X}}/K}^1 \xrightarrow{\gamma} \mathfrak{m}_{\mathbb{X}} \longrightarrow 0$$

is an exact sequence of graded  $R_{\mathbb{X}}$ -modules.

*Proof.* Let  $1 \leq m \leq n+1$ , let  $i \geq 0$ , and let  $\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_m} \in \Omega_{R_{\mathbb{X}}/K}^m$  with  $f \in (R_{\mathbb{X}})_i$  and  $0 \leq i_1 < \cdots < i_m \leq n$ . Then we have

$$(\gamma \circ d)(\omega) = i f dx_{i_1} \wedge \cdots \wedge dx_{i_m} - df \wedge \gamma(dx_{i_1} \wedge \cdots \wedge dx_{i_m})$$

and

$$\begin{aligned} (d \circ \gamma)(\omega) &= d(f \sum_{j=1}^m (-1)^{j+1} x_{i_j} dx_{i_1} \wedge \cdots \wedge \widehat{dx_{i_j}} \wedge \cdots \wedge dx_{i_m}) \\ &= df \wedge \gamma(dx_{i_1} \wedge \cdots \wedge dx_{i_m}) + m f dx_{i_1} \wedge \cdots \wedge dx_{i_m}. \end{aligned}$$

This implies  $(\gamma \circ d + d \circ \gamma)(\omega) = (m+i)\omega$ . Hence  $(\gamma \circ d + d \circ \gamma)(\omega) = \deg(\omega)\omega$  for every homogeneous element  $\omega \in \Omega_{R_{\mathbb{X}}/K}^m$ . Now suppose that  $\omega \in \Omega_{R_{\mathbb{X}}/K}^m \setminus \{0\}$  is a homogeneous element with  $\gamma(\omega) = 0$ . Set  $\tilde{\omega} = \frac{1}{\deg(\omega)} d\omega \in \Omega_{R_{\mathbb{X}}/K}^{m+1}$ . We get  $\gamma(\tilde{\omega}) = \omega$ , and the proof is complete.  $\square$

Obviously, the ring  $R_{\mathbb{X}}$  is Noetherian and the graded  $R_{\mathbb{X}}$ -module  $\Omega_{R_{\mathbb{X}}/K}^m$  is finitely generated, and so the **Hilbert polynomial** of  $\Omega_{R_{\mathbb{X}}/K}^m$  exists (cf. [KR2, 5.1.21]) and is denoted by  $\text{HP}_{\Omega_{R_{\mathbb{X}}/K}^m}(z)$ . The number  $\text{ri}(\Omega_{R_{\mathbb{X}}/K}^m) = \min\{i \in \mathbb{Z} \mid \text{HF}_{\Omega_{R_{\mathbb{X}}/K}^m}(j) = \text{HP}_{\Omega_{R_{\mathbb{X}}/K}^m}(j) \text{ for all } j \geq i\}$  is called the **regularity index** of  $\Omega_{R_{\mathbb{X}}/K}^m$ . In the following, we denote the Hilbert function of  $R_{\mathbb{X}}$  by  $\text{HF}_{\mathbb{X}}$  and its regularity index by  $r_{\mathbb{X}}$ . As a consequence of the exact sequence  $(\mathcal{K})$ , we have the following bound for  $\text{ri}(\Omega_{R_{\mathbb{X}}/K}^{n+1})$ .

**Corollary 2.5.** *We have  $\text{ri}(\Omega_{R_{\mathbb{X}}/K}^{n+1}) \leq \max\{r_{\mathbb{X}}, \text{ri}(\Omega_{R_{\mathbb{X}}/K}^1), \dots, \text{ri}(\Omega_{R_{\mathbb{X}}/K}^n)\}$ .*

Let us examine the Hilbert functions of the modules of Kähler differential  $m$ -forms and their regularity indices in a concrete case.

**Example 2.6.** Let  $\mathbb{X}$  and  $\mathbb{Y}$  be two sets of six reduced  $K$ -rational points in  $\mathbb{P}^2$  such that  $\mathbb{X}$  is contained in a non-singular conic and  $\mathbb{Y}$  lies on the union of two lines and no 5 points of  $\mathbb{Y}$  are collinear. Then the Hilbert functions of  $\mathbb{X}$  and  $\mathbb{Y}$  agree, as do the Hilbert functions of  $\Omega_{R_{\mathbb{X}}/K}^1$  and  $\Omega_{R_{\mathbb{Y}}/K}^1$ , namely

$$\begin{aligned} \text{HF}_{\mathbb{X}} &= \text{HF}_{\mathbb{Y}} & : 1 \ 3 \ 5 \ 6 \ 6 \ \cdots \\ \text{HF}_{\Omega_{R_{\mathbb{X}}/K}^1} &= \text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1} & : 0 \ 3 \ 8 \ 11 \ 10 \ 7 \ 6 \ 6 \ \cdots \end{aligned}$$

It is clear that  $r_{\mathbb{X}} = r_{\mathbb{Y}} = 3$  and  $\text{ri}(\Omega_{R_{\mathbb{X}}/K}^1) = \text{ri}(\Omega_{R_{\mathbb{Y}}/K}^1) = 6$ . We also have  $\text{HF}_{\Omega_{R_{\mathbb{X}}/K}^m}(i) = \text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^m}(i) = 0$  for  $m = 1, 2, 3$  and  $i \leq 0$ .

However, the Hilbert functions of  $\Omega_{R_{\mathbb{X}}/K}^2$  and  $\Omega_{R_{\mathbb{Y}}/K}^2$  are different, and also the Hilbert functions of  $\Omega_{R_{\mathbb{X}}/K}^3$  and  $\Omega_{R_{\mathbb{Y}}/K}^3$  disagree:

$$\begin{aligned} \text{HF}_{\Omega_{R_{\mathbb{X}}/K}^2} &: 0 \ 0 \ 3 \ 6 \ 4 \ 1 \ 0 \ 0 \cdots, & \text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^2} &: 0 \ 0 \ 3 \ 6 \ 5 \ 1 \ 0 \ 0 \cdots, \\ \text{HF}_{\Omega_{R_{\mathbb{X}}/K}^3} &: 0 \ 0 \ 0 \ 1 \ 0 \ 0 \cdots, & \text{HF}_{\Omega_{R_{\mathbb{Y}}/K}^3} &: 0 \ 0 \ 0 \ 1 \ 1 \ 0 \cdots. \end{aligned}$$

In addition, we have  $\text{ri}(\Omega_{R_{\mathbb{X}}/K}^2) = \text{ri}(\Omega_{R_{\mathbb{Y}}/K}^2) = 6$ ,  $\text{ri}(\Omega_{R_{\mathbb{X}}/K}^3) = 4$ , and  $\text{ri}(\Omega_{R_{\mathbb{Y}}/K}^3) = 5$ . In this case, the inequality of regularity indices in Corollary 2.5 is a strict inequality. Moreover, the Hilbert functions of the exterior powers  $\Omega_{R_{\mathbb{X}}/K}^2$  and  $\Omega_{R_{\mathbb{X}}/K}^3$  distinguish a set  $\mathbb{X}$  of six points on an irreducible conic from a set  $\mathbb{Y}$  of six points on a reducible conic.

### 3. KÄHLER DIFFERENTIAL ALGEBRAS FOR SUBSCHEMES OF $\mathbb{P}^1$

In this section we consider the easiest case, namely 0-dimensional subschemes  $\mathbb{X}$  of  $\mathbb{P}^1$ . It is well known that Hilbert functions do not change under base field extensions (for instance, see [KR2, 5.1.20]). Thus, in order to compute the Hilbert function of the Kähler differential algebra for the 0-dimensional scheme  $\mathbb{X}$  of  $\mathbb{P}^1$ , we may assume that the field  $K$  is algebraically closed. In this case the homogeneous vanishing ideal  $I_{\mathbb{X}}$  is a principal ideal generated by a homogeneous polynomial  $F \in S = K[X_0, X_1]$ . Moreover, after a suitable change of coordinates, we may also assume that  $F$  is of the form  $F = \prod_{i=1}^s (X_1 - a_i X_0)^{m_i}$  where  $s \geq 1$ ,  $m_1, \dots, m_s \geq 1$  and  $a_1, \dots, a_s \in K$  such that  $a_i \neq a_j$  for  $i \neq j$ .

In [Rob, Section 4], L.G. Roberts gave a formula for the Hilbert function of  $\Omega_{R_{\mathbb{X}}/K}$  when  $m_1 = m_2 = \dots = m_s = 1$ . Now we extend his result to arbitrary exponents  $m_1, \dots, m_s \geq 1$  as follows.

**Proposition 3.1.** *Let  $\mathbb{X} \subseteq \mathbb{P}^1$  be a 0-dimensional scheme, and let  $I_{\mathbb{X}} = \langle F \rangle$ , where  $F = \prod_{i=1}^s (X_1 - a_i X_0)^{m_i}$  for some  $s$ ,  $m_1, \dots, m_s \geq 1$ , and  $a_i \in K$  with  $a_i \neq a_j$  for  $i \neq j$ , and let  $\mu = \sum_{i=1}^s m_i$ . Then the Hilbert functions of the Kähler differential modules of  $R_{\mathbb{X}}/K$  are given by*

$$\begin{aligned} \text{HF}_{\Omega_{R_{\mathbb{X}}/K}^1} &: 0 \ 2 \ 4 \ 6 \ \cdots \ 2(\mu-2) \ 2(\mu-1) \ 2\mu-1 \ 2\mu-2 \ \cdots \ 2\mu-s \ 2\mu-s \cdots \\ \text{HF}_{\Omega_{R_{\mathbb{X}}/K}^2} &: 0 \ 0 \ 1 \ 2 \ \cdots \ \mu-2 \ \mu-1 \ \mu-2 \ \mu-3 \ \cdots \ \mu-s \ \mu-s \cdots \end{aligned}$$

In particular, we have  $\text{ri}(\Omega_{R_{\mathbb{X}}/K}^1) = \text{ri}(\Omega_{R_{\mathbb{X}}/K}^2) = \mu + s - 1$ .

*Proof.* Let  $G = \prod_{i=1}^s (X_1 - a_i X_0)^{m_i-1}$ , let  $H_1 = \sum_{i=1}^s m_i a_i \prod_{j \neq i} (X_1 - a_j X_0)$ , and let  $H_2 = \sum_{i=1}^s m_i \prod_{j \neq i} (X_1 - a_j X_0)$ . Note that  $\deg(G) = \sum_{i=1}^s (m_i - 1)$  and  $\deg(H_1) = \deg(H_2) = s - 1$ . We verify that  $\gcd(H_1, H_2) = 1$ . Suppose for a contradiction that  $\gcd(H_1, H_2) = H$  with  $\deg(H) \geq 1$ . Euler's relation  $\mu F = G(-X_0 H_1 + X_1 H_2)$  implies  $\mu \prod_{i=1}^s (X_1 - a_i X_0) = -X_0 H_1 + X_1 H_2$ . So,  $H$  is a divisor of  $\prod_{i=1}^s (X_1 - a_i X_0)$ . There exists an index  $i \in \{1, \dots, s\}$  such that  $(X_1 - a_i X_0) \mid H$ , but  $(X_1 - a_i X_0) \nmid H_2$ , a contradiction. Hence we obtain  $\gcd(H_1, H_2) = 1$ . Thus the sequence  $\{H_1, H_2\}$  is an  $S$ -regular sequence. Consequently, this sequence is also a regular sequence for the principal ideal  $\langle G \rangle$  which is regarded as a graded

$S$ -module. So, for  $i \in \mathbb{Z}$ , we have

$$\begin{aligned}
\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^2}(i) &= \mathrm{HF}_{S/\langle \frac{\partial F}{\partial X_0}, \frac{\partial F}{\partial X_1} \rangle}(i-2) \\
&= \mathrm{HF}_{S/\langle GH_1, GH_2 \rangle}(i-2) \\
&= \mathrm{HF}_{S/\langle G \rangle}(i-2) + \mathrm{HF}_{\langle G \rangle/\langle GH_1, GH_2 \rangle}(i-2) \\
&= \mathrm{HF}_{S/\langle G \rangle}(i-2) + \mathrm{HF}_{\langle G \rangle}(i-2) - 2\mathrm{HF}_{\langle G \rangle}(i-1-s) + \mathrm{HF}_{\langle G \rangle}(i-2s) \\
&= \mathrm{HF}_S(i-2) - 2\mathrm{HF}_S(i-1-\mu) + \mathrm{HF}_S(i-s-\mu) \\
&= \binom{i-1}{1} - 2\binom{i-\mu}{1} + \binom{i-s-\mu+1}{1}
\end{aligned}$$

Thus the Hilbert function of  $\Omega_{R_{\mathbb{X}}/K}^2$  is

$$\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^2} : 0 \ 0 \ 1 \ 2 \ \cdots \ \mu-2 \ \mu-1 \ \mu-2 \ \mu-3 \ \cdots \ \mu-s \ \mu-s \ \cdots .$$

Moreover, it is clear that  $\mathrm{HF}_{\mathfrak{m}_{\mathbb{X}}} : 0 \ 2 \ 3 \ 4 \ \cdots \ \mu-1 \ \mu \ \mu \ \cdots$ . By Proposition 2.4, we have the exact sequence of graded  $R_{\mathbb{X}}$ -modules

$$0 \longrightarrow \Omega_{R_{\mathbb{X}}/K}^2 \longrightarrow \Omega_{R_{\mathbb{X}}/K}^1 \longrightarrow \mathfrak{m}_{\mathbb{X}} \longrightarrow 0.$$

Hence the Hilbert function of  $\Omega_{R_{\mathbb{X}}/K}^1$  satisfies  $\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^1}(i) = \mathrm{HF}_{\mathfrak{m}_{\mathbb{X}}}(i) + \mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^2}(i)$  for all  $i \in \mathbb{Z}$ . More precisely, it is of the form

$$\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^1} : 0 \ 2 \ 4 \ 6 \ \cdots \ 2(\mu-2) \ 2(\mu-1) \ 2\mu-1 \ 2\mu-2 \ \cdots \ 2\mu-s \ 2\mu-s \ \cdots$$

as claimed.  $\square$

Let us apply this proposition in an explicit example.

**Example 3.2.** Let  $\mathbb{X} \subseteq \mathbb{P}^1$  be the 0-dimensional scheme with the homogeneous vanishing ideal  $I_{\mathbb{X}} = \langle X_1(X_1 - X_0)^2(X_1 - 2X_0)^3 \rangle$ . Clearly, we have  $s = 3$  and  $\mu = 6$  and  $\mathrm{HF}_{\mathfrak{m}_{\mathbb{X}}} : 0 \ 2 \ 3 \ 4 \ 5 \ 6 \ 6 \ \cdots$ . An application of Proposition 3.1 yields

$$\begin{aligned}
\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^1} &: 0 \ 2 \ 4 \ 6 \ 8 \ 10 \ 11 \ 10 \ 9 \ 9 \ \cdots \\
\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^2} &: 0 \ 0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 4 \ 3 \ 3 \ \cdots
\end{aligned}$$

In this case we have  $\mathrm{ri}(\Omega_{R_{\mathbb{X}}/K}^1) = \mathrm{ri}(\Omega_{R_{\mathbb{X}}/K}^2) = \mu + s - 1 = 8$ .

#### 4. SPECIAL VALUES OF THE HILBERT FUNCTION OF $\Omega_{R_{\mathbb{X}}/K}^m$

In this section we describe the values of the Hilbert function of the module of Kähler differential  $m$ -forms for a 0-dimensional scheme  $\mathbb{X}$  at some special degrees. From now on, the coordinates  $\{X_0, \dots, X_n\}$  of  $\mathbb{P}^n$  are always chosen such that no point of  $\mathbb{X}$  lies on the hyperplane  $\mathcal{Z}^+(X_0)$ . By the choice of the coordinates,  $x_0$  is a non-zerodivisor of  $R_{\mathbb{X}}$ . Moreover,  $x_0$  is also a non-zerodivisor for any nontrivial graded submodule of a graded free  $R_{\mathbb{X}}$ -module, as the following lemma shows.

**Lemma 4.1.** *Let  $d \geq 1$ , let  $\delta_1, \dots, \delta_d \in \mathbb{Z}$ , and let  $V$  be a non-trivial graded submodule of the graded free  $R_{\mathbb{X}}$ -module  $\bigoplus_{j=1}^d R_{\mathbb{X}}(-\delta_j)$ . Then  $x_0$  is not a zerodivisor for  $V$ , i.e., if  $x_0 \cdot v = 0$  for some  $v \in V$  then  $v = 0$ .*

*Proof.* Let  $\{e_1, \dots, e_d\}$  be the canonical  $R_{\mathbb{X}}$ -basis of  $\bigoplus_{j=1}^d R_{\mathbb{X}}(-\delta_j)$ , and let  $i \in \mathbb{Z}$ . Then every homogeneous element  $v \in V_i$  has a representation  $v = g_1 e_1 + \dots + g_d e_d$  for some homogeneous elements  $g_1, \dots, g_d \in R_{\mathbb{X}}$ , where  $\deg(g_j) = \deg(v) - \delta_j$  for  $j = 1, \dots, d$ . Suppose that  $x_0 \cdot v = 0$ . This implies that  $x_0 g_1 e_1 + \dots + x_0 g_d e_d = 0$ , and so  $x_0 g_1 = \dots = x_0 g_d = 0$  in  $R_{\mathbb{X}}$ . Since  $x_0$  is a non-zero-divisor for  $R_{\mathbb{X}}$ , we have  $g_1 = \dots = g_d = 0$ , and hence  $v = 0$ . Thus the claim follows.  $\square$

The number  $\alpha_{\mathbb{X}} = \min\{i \in \mathbb{N} \mid (I_{\mathbb{X}})_i \neq 0\}$  is called the **initial degree** of  $I_{\mathbb{X}}$ . Using this notation, some basic properties of  $\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^m}$  can be summarized as follows.

**Proposition 4.2.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme, and let  $1 \leq m \leq n+1$ .*

- (a) *For  $i < m$ , we have  $\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^m}(i) = 0$ .*
- (b) *For  $m \leq i < \alpha_{\mathbb{X}} + m - 1$ , we have  $\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^m}(i) = \binom{n+1}{m} \cdot \binom{n+i-m}{n}$ .*
- (c) *The Hilbert polynomial of  $\Omega_{R_{\mathbb{X}}/K}^m$  is constant.*
- (d) *We have  $\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^m}(r_{\mathbb{X}}+m) \geq \mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^m}(r_{\mathbb{X}}+m+1) \geq \dots$ , and if  $\mathrm{ri}(\Omega_{R_{\mathbb{X}}/K}^m) \geq r_{\mathbb{X}} + m$  then*

$$\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^m}(r_{\mathbb{X}} + m) > \mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^m}(r_{\mathbb{X}} + m + 1) > \dots > \mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^m}(\mathrm{ri}(\Omega_{R_{\mathbb{X}}/K}^m)).$$

*Proof.* (a) Obviously, every non-zero homogeneous element  $\omega$  of  $\Omega_{R_{\mathbb{X}}/K}^m$  has degree  $\deg(\omega) \geq m$ , and hence  $\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^m}(i) = 0$  for all  $i < m$ .

(b) Let  $m \leq i < \alpha_{\mathbb{X}} + m - 1$ . Notice that  $I_{\mathbb{X}} \Omega_{S/K}^m \subseteq dI_{\mathbb{X}} \Omega_{S/K}^{m-1}$ . Also, we have  $(dI_{\mathbb{X}} \Omega_{S/K}^{m-1})_i = \langle 0 \rangle$  for all  $i < \alpha_{\mathbb{X}} + m - 1$ , since a non-zero homogeneous element of  $dI_{\mathbb{X}} \Omega_{S/K}^{m-1}$  is always of the form  $\sum_k dF_k \wedge \omega_k$ , where  $F_k \in (I_{\mathbb{X}})_{\geq \alpha_{\mathbb{X}}}$  and  $\omega_k \in (\Omega_{S/K}^{m-1})_{\geq m-1}$ . By Proposition 2.2, for all  $i < \alpha_{\mathbb{X}} + m - 1$ , we obtain

$$\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^m}(i) = \mathrm{HF}_{\Omega_{S/K}^m}(i) = \binom{n+1}{m} \cdot \binom{n+i-m}{n}.$$

(c) It follows from Proposition 2.2 that

$$\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^m}(i) \leq \mathrm{HF}_{\Omega_{S/K}^m/I_{\mathbb{X}} \Omega_{S/K}^m}(i) = \binom{n+1}{m} \mathrm{HF}_{\mathbb{X}}(i) \leq \binom{n+1}{m} \deg(\mathbb{X})$$

for all  $i \in \mathbb{Z}$ . Hence the Hilbert polynomial of  $\Omega_{R_{\mathbb{X}}/K}^m$  is a constant polynomial.

(d) The graded  $R_{\mathbb{X}}$ -module  $\Omega_{R_{\mathbb{X}}/K}^m$  has the following form:

$$(\Omega_{R_{\mathbb{X}}/K}^m)_{i+m} = (R_{\mathbb{X}})_i dx_0 \wedge \dots \wedge dx_{m-1} + \dots + (R_{\mathbb{X}})_i dx_{n-m+1} \wedge \dots \wedge dx_n.$$

Observe that  $(R_{\mathbb{X}})_i = x_0 (R_{\mathbb{X}})_{i-1}$  if  $i > r_{\mathbb{X}}$ . Thus  $(\Omega_{R_{\mathbb{X}}/K}^m)_{i+m} = x_0 (\Omega_{R_{\mathbb{X}}/K}^m)_{i+m-1}$  for all  $i > r_{\mathbb{X}}$ . So, for all  $i > r_{\mathbb{X}}$ , we have the inequality

$$\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^m}(i+m-1) \geq \mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^m}(i+m).$$

Now let  $\mathcal{G} = \langle (\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n}) \mid F \in I_{\mathbb{X}} \rangle_{R_{\mathbb{X}}}$ . By [DK, Proposition 1.3] and [SS, X.83], there is an exact sequence of graded  $R_{\mathbb{X}}$ -modules

$$0 \longrightarrow \mathcal{G} \wedge_{R_{\mathbb{X}}} \bigwedge_{R_{\mathbb{X}}}^{m-1} (R_{\mathbb{X}}^{n+1}) \longrightarrow \bigwedge_{R_{\mathbb{X}}}^m (R_{\mathbb{X}}^{n+1}) \longrightarrow \Omega_{R_{\mathbb{X}}/K}^m(m) \longrightarrow 0.$$



Suppose  $i \geq r_{\mathbb{X}}$  satisfies  $\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^m}(i+m) = \mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^m}(i+m+1)$ . Then it follows from the above exact sequence that

$$\begin{aligned} \mathrm{HF}_{\wedge_{R_{\mathbb{X}}}^m(R_{\mathbb{X}}^{n+1})}(i) - \mathrm{HF}_{\mathcal{G} \wedge_{R_{\mathbb{X}}} \wedge_{R_{\mathbb{X}}}^{m-1}(R_{\mathbb{X}}^{n+1})}(i) \\ = \mathrm{HF}_{\wedge_{R_{\mathbb{X}}}^m(R_{\mathbb{X}}^{n+1})}(i+1) - \mathrm{HF}_{\mathcal{G} \wedge_{R_{\mathbb{X}}} \wedge_{R_{\mathbb{X}}}^{m-1}(R_{\mathbb{X}}^{n+1})}(i+1). \end{aligned}$$

For every  $j \geq r_{\mathbb{X}}$ ,  $\mathrm{HF}_{\mathbb{X}}(j) = \deg(\mathbb{X})$ , and so  $\mathrm{HF}_{\wedge_{R_{\mathbb{X}}}^m(R_{\mathbb{X}}^{n+1})}(j) = \mathrm{HF}_{\wedge_{R_{\mathbb{X}}}^m(R_{\mathbb{X}}^{n+1})}(j+1)$ . Consequently, we have

$$\mathrm{HF}_{\mathcal{G} \wedge_{R_{\mathbb{X}}} \wedge_{R_{\mathbb{X}}}^{m-1}(R_{\mathbb{X}}^{n+1})}(i) = \mathrm{HF}_{\mathcal{G} \wedge_{R_{\mathbb{X}}} \wedge_{R_{\mathbb{X}}}^{m-1}(R_{\mathbb{X}}^{n+1})}(i+1).$$

In addition, Lemma 4.1 shows that  $x_0$  is a non-zerodivisor for the graded  $R_{\mathbb{X}}$ -submodule  $\mathcal{G} \wedge_{R_{\mathbb{X}}} \wedge_{R_{\mathbb{X}}}^{m-1}(R_{\mathbb{X}}^{n+1})$  of the graded-free  $R_{\mathbb{X}}$ -module  $\wedge_{R_{\mathbb{X}}}^m(R_{\mathbb{X}}^{n+1})$ . This implies

$$(\mathcal{G} \wedge_{R_{\mathbb{X}}} \wedge_{R_{\mathbb{X}}}^{m-1}(R_{\mathbb{X}}^{n+1}))_{i+1} = x_0(\mathcal{G} \wedge_{R_{\mathbb{X}}} \wedge_{R_{\mathbb{X}}}^{m-1}(R_{\mathbb{X}}^{n+1}))_i.$$

In view of [GM, Proposition 1.1], the ideal  $I_{\mathbb{X}}$  can be generated by homogeneous polynomials of degrees  $\leq r_{\mathbb{X}} + 1$ . So, the graded  $R_{\mathbb{X}}$ -module  $\mathcal{G} \wedge_{R_{\mathbb{X}}} \wedge_{R_{\mathbb{X}}}^{m-1}(R_{\mathbb{X}}^{n+1})$  is generated in degrees  $\leq r_{\mathbb{X}}$ . Thus we obtain

$$\begin{aligned} (\mathcal{G} \wedge_{R_{\mathbb{X}}} \wedge_{R_{\mathbb{X}}}^{m-1}(R_{\mathbb{X}}^{n+1}))_{i+2} \\ = x_0(\mathcal{G} \wedge_{R_{\mathbb{X}}} \wedge_{R_{\mathbb{X}}}^{m-1}(R_{\mathbb{X}}^{n+1}))_{i+1} + \cdots + x_n(\mathcal{G} \wedge_{R_{\mathbb{X}}} \wedge_{R_{\mathbb{X}}}^{m-1}(R_{\mathbb{X}}^{n+1}))_{i+1} \\ = x_0(x_0(\mathcal{G} \wedge_{R_{\mathbb{X}}} \wedge_{R_{\mathbb{X}}}^{m-1}(R_{\mathbb{X}}^{n+1}))_i + \cdots + x_n(\mathcal{G} \wedge_{R_{\mathbb{X}}} \wedge_{R_{\mathbb{X}}}^{m-1}(R_{\mathbb{X}}^{n+1}))_i) \\ = x_0(\mathcal{G} \wedge_{R_{\mathbb{X}}} \wedge_{R_{\mathbb{X}}}^{m-1}(R_{\mathbb{X}}^{n+1}))_{i+1}. \end{aligned}$$

Altogether, we have  $\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^m}(i+m+1) = \mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^m}(i+m+2)$ , and the claim follows by induction.  $\square$

The following example shows that  $\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^m}(i)$  may or may not be monotonic in the range  $\alpha_{\mathbb{X}} + m \leq i \leq r_{\mathbb{X}} + m$ .

**Example 4.3.** Let  $K = \mathbb{Q}$ , and let  $\mathbb{X} \subseteq \mathbb{P}^2$  be the set of nine points  $\mathbb{X} = \{(1 : 1 : 0), (1 : 1 : 1), (1 : 1 : 2), (1 : 1 : 3), (1 : 1 : 4), (1 : 1 : 5), (1 : 0 : 1), (1 : 2 : 1), (1 : 2 : 2)\}$ . Notice that  $\mathbb{X}$  contains six points on a line and three non-collinear points off that line. It is clear that  $\mathrm{HF}_{\mathbb{X}} : 1 \ 3 \ 6 \ 7 \ 8 \ 9 \ 9 \cdots$ ,  $\alpha_{\mathbb{X}} = 3$ , and  $r_{\mathbb{X}} = 5$ . The Hilbert functions of the Kähler differential modules of  $R_{\mathbb{X}}/K$  are given by

$$\begin{aligned} \mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^1} &: 0 \ 3 \ 9 \ 15 \ 14 \ 13 \ 14 \ 13 \ 12 \ 11 \ 10 \ 9 \ 9 \cdots \\ \mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^2} &: 0 \ 0 \ 3 \ 9 \ 9 \ 4 \ 5 \ 4 \ 3 \ 2 \ 1 \ 0 \ 0 \cdots \\ \mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^3} &: 0 \ 0 \ 0 \ 1 \ 3 \ 0 \ 0 \cdots \end{aligned}$$

We see that  $\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^1}(\alpha_{\mathbb{X}}+1) = 14 > 13 = \mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^1}(\alpha_{\mathbb{X}}+2)$  and  $\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^1}(\alpha_{\mathbb{X}}+2) = 13 < 14 = \mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^1}(r_{\mathbb{X}}+1)$ . So,  $\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^1}(i)$  is not monotonic in the range  $\alpha_{\mathbb{X}}+1 \leq i \leq r_{\mathbb{X}}+1$ . Similarly,  $\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^2}(i)$  is not monotonic in the range  $\alpha_{\mathbb{X}}+2 \leq i \leq r_{\mathbb{X}}+2$ .

Next we consider the set  $\mathbb{Y} = \mathbb{X} \cup \{(1 : 0 : 2)\}$ . We have  $\mathrm{HF}_{\mathbb{Y}} : 1 \ 3 \ 6 \ 8 \ 9 \ 10 \ 10 \cdots$ ,  $\alpha_{\mathbb{Y}} = 3$ , and  $r_{\mathbb{Y}} = 5$ . The Hilbert functions of the Kähler differential modules

of  $R_{\mathbb{Y}}/K$  are

$$\begin{aligned} \mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1} &: 0 \ 3 \ 9 \ 16 \ 18 \ 16 \ 15 \ 14 \ 13 \ 12 \ 11 \ 10 \ 10 \cdots \\ \mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^2} &: 0 \ 0 \ 3 \ 9 \ 12 \ 8 \ 5 \ 4 \ 3 \ 2 \ 1 \ 0 \ 0 \cdots \\ \mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^3} &: 0 \ 0 \ 0 \ 1 \ 3 \ 2 \ 0 \cdots \end{aligned}$$

Hence  $\mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^1}(i)$  is monotonic in the range  $\alpha_{\mathbb{Y}} + 1 \leq i \leq r_{\mathbb{Y}} + 1$ , and  $\mathrm{HF}_{\Omega_{R_{\mathbb{Y}}/K}^2}(i)$  is also monotonic in the range  $\alpha_{\mathbb{Y}} + 2 \leq i \leq r_{\mathbb{Y}} + 2$ .

## 5. BOUNDS FOR THE REGULARITY INDEX OF $\Omega_{R_{\mathbb{X}}/K}^m$

In this section we give an upper bound for the regularity index of the module of Kähler differential  $m$ -forms  $\Omega_{R_{\mathbb{X}}/K}^m$  for a 0-dimensional scheme  $\mathbb{X}$  in  $\mathbb{P}^n$ . To do this, we need the following lemmas.

**Lemma 5.1.** *Let  $d \geq 1$ , let  $\delta_1, \dots, \delta_d \in \mathbb{Z}$  such that  $\delta_1 \leq \dots \leq \delta_d$ , let  $W = \bigoplus_{j=1}^d R_{\mathbb{X}}(-\delta_j)$  be the graded free  $R_{\mathbb{X}}$ -module, and let  $V$  be a non-trivial graded submodule of  $W$ . Then, for  $1 \leq m \leq d$ , we have*

$$\mathrm{ri}(V \wedge_{R_{\mathbb{X}}} \bigwedge_{R_{\mathbb{X}}}^m(W)) \leq \mathrm{ri}(V) + \delta_{d-m+1} + \dots + \delta_d.$$

*Proof.* First we note that the Hilbert polynomial of  $W$  is  $\mathrm{HP}_W(z) = d \cdot \deg(\mathbb{X})$  and that  $\mathrm{ri}(W) = r_{\mathbb{X}} + \delta_d$ . This shows that the Hilbert polynomial of  $V$  is a constant polynomial  $\mathrm{HP}_V(z) = u \leq d \cdot \deg(\mathbb{X})$ . Let  $r = \mathrm{ri}(V)$ , and let  $v_1, \dots, v_u$  be a  $K$ -basis of  $V_r$ . By Lemma 4.1, the elements  $\{x_0^i v_1, \dots, x_0^i v_u\}$  form a  $K$ -basis of the  $K$ -vector space  $V_{r+i}$  for all  $i \in \mathbb{N}$ . We let  $\{e_1, \dots, e_d\}$  be the canonical  $R_{\mathbb{X}}$ -basis of  $W$ , we let  $t = \binom{d}{m}$ , and we let  $\{\varepsilon_1, \dots, \varepsilon_t\}$  be a basis of the graded free  $R_{\mathbb{X}}$ -module  $\bigwedge_{R_{\mathbb{X}}}^m(W)$  w.r.t.  $\{e_1, \dots, e_d\}$ . We set  $\delta = \delta_{d-m+1} + \dots + \delta_d$ , and let

$$N = \langle x_0^{\delta - \deg(\varepsilon_k)} v_j \wedge \varepsilon_k \in V \wedge_{R_{\mathbb{X}}} \bigwedge_{R_{\mathbb{X}}}^m(W) \mid 1 \leq j \leq u, 1 \leq k \leq t \rangle_K.$$

Let  $\varrho = \dim_K N$ , and let  $w_1, \dots, w_{\varrho}$  be a  $K$ -basis of  $N$ . It is not difficult to verify that  $N = \langle w_1, \dots, w_{\varrho} \rangle_K = (V \wedge_{R_{\mathbb{X}}} \bigwedge_{R_{\mathbb{X}}}^m(W))_{\delta+r}$ . Moreover, for any  $i \geq 0$ , the set  $\{x_0^i w_1, \dots, x_0^i w_{\varrho}\}$  is  $K$ -linearly independent. Indeed, assume that there are elements  $a_1, \dots, a_{\varrho} \in K$  such that  $\sum_{j=1}^{\varrho} x_0^i a_j w_j = 0$ . Since  $x_0$  is a non-zero divisor for  $V \wedge_{R_{\mathbb{X}}} \bigwedge_{R_{\mathbb{X}}}^m(W)$  by Lemma 4.1, we get  $\sum_{j=1}^{\varrho} a_j w_j = 0$ , and hence  $a_1 = \dots = a_{\varrho} = 0$ .

Now it suffices to prove that the set  $\{x_0^i w_1, \dots, x_0^i w_{\varrho}\}$  generates the  $K$ -vector space  $(V \wedge_{R_{\mathbb{X}}} \bigwedge_{R_{\mathbb{X}}}^m(W))_{\delta+r+i}$  for all  $i \geq 0$ . Let  $w \in (V \wedge_{R_{\mathbb{X}}} \bigwedge_{R_{\mathbb{X}}}^m(W))_{\delta+r+i}$  be a non-zero homogeneous element. Then  $w = \sum_{j,k} \tilde{v}_j \wedge h_k \varepsilon_k = \sum_{j,k} h_k \tilde{v}_j \wedge \varepsilon_k$  for some homogeneous elements  $\tilde{v}_j \in V$  and  $h_k \in R_{\mathbb{X}}$  such that  $\deg(\tilde{v}_j) + \deg(h_k) = \delta + r + i - \deg(\varepsilon_k)$  for all  $j, k$ . Note that  $\deg(h_k \tilde{v}_j) = \delta + r + i - \deg(\varepsilon_k) \geq r + i$ . Also, we have

$$h_k \tilde{v}_j \in V_{\delta+r+i-\deg(\varepsilon_k)} = \langle x_0^{\delta+i-\deg(\varepsilon_k)} v_1, \dots, x_0^{\delta+i-\deg(\varepsilon_k)} v_u \rangle_K.$$

So, there are  $b_{jk1}, \dots, b_{jku} \in K$  such that  $h_k \tilde{v}_j = \sum_{l=1}^u b_{jkl} x_0^{\delta+i-\deg(\varepsilon_k)} v_l$ . This implies

$$\begin{aligned} w &= \sum_{j,k} h_k \tilde{v}_j \wedge \varepsilon_k = \sum_{j,k} \sum_{l=1}^u b_{jkl} x_0^{\delta+i-\deg(\varepsilon_k)} v_l \wedge \varepsilon_k \\ &= \sum_{j,k} \sum_{l=1}^u b_{jkl} x_0^i (x_0^{\delta-\deg(\varepsilon_k)} v_l \wedge \varepsilon_k) = \sum_{j,k} \sum_{l=1}^u \sum_{q=1}^{\varrho} b_{jkl} c_{jklq} x_0^i w_q \end{aligned}$$

for some  $c_{jklq} \in K$ . Thus we get  $w \in \langle x_0^i w_1, \dots, x_0^i w_{\varrho} \rangle_K$ , and consequently  $\mathrm{HF}_{V \wedge_{R_{\mathbb{X}}} \bigwedge_{R_{\mathbb{X}}}^m(W)}(i) = \varrho$  for all  $i \geq \delta + r$ . Therefore  $\mathrm{ri}(V \wedge_{R_{\mathbb{X}}} \bigwedge_{R_{\mathbb{X}}}^m(W)) \leq \mathrm{ri}(V) + \delta$ , as we wanted to show.  $\square$

**Lemma 5.2.** *Let  $V$  be a graded  $R_{\mathbb{X}}$ -module generated by the set of homogeneous elements  $\{v_1, \dots, v_d\}$  for some  $d \geq 1$ . Let  $\delta_j = \deg(v_j)$  for  $j = 1, \dots, d$ , and let  $m \geq 1$ . Assume that  $\delta_1 \leq \dots \leq \delta_d$ , and set  $\delta = \delta_{d-m+1} + \dots + \delta_d$  if  $m \leq d$ . Then the regularity index of  $\bigwedge_{R_{\mathbb{X}}}^m(V)$  satisfies  $\mathrm{ri}(\bigwedge_{R_{\mathbb{X}}}^m(V)) = -\infty$  if  $m > d$  and*

$$\mathrm{ri}(\bigwedge_{R_{\mathbb{X}}}^m(V)) \leq \max \{ r_{\mathbb{X}} + \delta + \delta_d - \delta_{d-m+1}, \mathrm{ri}(V) + \delta - \delta_{d-m+1} \}$$

if  $1 \leq m \leq d$ . In particular, if  $1 \leq m \leq d$  and  $\delta_1 = \dots = \delta_d = t$  then we have  $\mathrm{ri}(\bigwedge_{R_{\mathbb{X}}}^m(V)) \leq \max \{ r_{\mathbb{X}} + mt, \mathrm{ri}(V) + (m-1)t \}$ .

*Proof.* If  $m > d$ , then  $\bigwedge_{R_{\mathbb{X}}}^m(V) = \langle 0 \rangle$ , and hence  $\mathrm{ri}(\bigwedge_{R_{\mathbb{X}}}^m(V)) = -\infty$ . Now we assume that  $1 \leq m \leq d$ . Obviously, the  $R_{\mathbb{X}}$ -linear map  $\alpha : W = \bigoplus_{j=1}^d R_{\mathbb{X}}(-\delta_j) \rightarrow V$  given by  $e_j \mapsto v_j$  is a homogeneous  $R_{\mathbb{X}}$ -epimorphism of degree zero. Set  $\mathcal{G} = \mathrm{Ker}(\alpha)$ . According to [SS, X.83], there is an exact sequence of graded  $R_{\mathbb{X}}$ -modules

$$0 \longrightarrow \mathcal{G} \wedge_{R_{\mathbb{X}}} \bigwedge_{R_{\mathbb{X}}}^{m-1}(W) \longrightarrow \bigwedge_{R_{\mathbb{X}}}^m(W) \xrightarrow{\bigwedge^m(\alpha)} \bigwedge_{R_{\mathbb{X}}}^m(V) \longrightarrow 0.$$

Thus an application of Lemma 5.1 yields that

$$\begin{aligned} \mathrm{ri}(\bigwedge_{R_{\mathbb{X}}}^m(V)) &\leq \max \{ \mathrm{ri}(\bigwedge_{R_{\mathbb{X}}}^m(W)), \mathrm{ri}(\mathcal{G} \wedge_{R_{\mathbb{X}}} \bigwedge_{R_{\mathbb{X}}}^{m-1}(W)) \} \\ &\leq \max \{ r_{\mathbb{X}} + \delta, \mathrm{ri}(\mathcal{G}) + \delta - \delta_{d-m+1} \} \\ &\leq \max \{ r_{\mathbb{X}} + \delta + \delta_d - \delta_{d-m+1}, \mathrm{ri}(V) + \delta - \delta_{d-m+1} \}. \end{aligned}$$

Here the last inequality follows from the fact that  $\mathrm{ri}(\mathcal{G}) \leq \max \{ r_{\mathbb{X}} + \delta_d, \mathrm{ri}(V) \}$ .  $\square$

Now we are able to give an upper bound for the regularity index of the module of Kähler differential  $m$ -forms  $\Omega_{R_{\mathbb{X}}/K}^m$ .

**Proposition 5.3.** *Let  $\mathbb{X} \subseteq \mathbb{P}^n$  be a 0-dimensional scheme, and let  $1 \leq m \leq n+1$ . The regularity index of the module of Kähler differential  $m$ -forms  $\Omega_{R_{\mathbb{X}}/K}^m$  satisfies*

$$\mathrm{ri}(\Omega_{R_{\mathbb{X}}/K}^m) \leq \max \{ r_{\mathbb{X}} + m, \mathrm{ri}(\Omega_{R_{\mathbb{X}}/K}^1) + m - 1 \}.$$

*Proof.* We set  $\mathcal{G} = \langle (\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n}) \in R_{\mathbb{X}}^{n+1} \mid F \in I_{\mathbb{X}} \rangle$ . By [DK, Proposition 1.3], we have the short exact sequence of graded  $R_{\mathbb{X}}$ -modules

$$0 \longrightarrow \mathcal{G}(-1) \longrightarrow R_{\mathbb{X}}^{n+1}(-1) \longrightarrow \Omega_{R_{\mathbb{X}}/K}^1 \longrightarrow 0.$$

Applying Lemma 5.2 to the graded  $R_{\mathbb{X}}$ -module  $\Omega_{R_{\mathbb{X}}/K}^1$  which is generated by the set  $\{dx_0, \dots, dx_n\}$ , we get  $\mathrm{ri}(\Omega_{R_{\mathbb{X}}/K}^m) \leq \max \{ r_{\mathbb{X}} + m, \mathrm{ri}(\Omega_{R_{\mathbb{X}}/K}^1) + m - 1 \}$ , as we wished.  $\square$

**Remark 5.4.** We have  $\text{ri}(\Omega_{R_{\mathbb{X}}/K}^{n+1}) \leq \max\{r_{\mathbb{X}} + n, \text{ri}(\Omega_{R_{\mathbb{X}}/K}^1) + n - 1\}$ . Indeed, the exact sequence  $(\mathcal{K})$  of graded  $R_{\mathbb{X}}$ -modules yields

$$\text{ri}(\Omega_{R_{\mathbb{X}}/K}^{n+1}) \leq \max\{\text{ri}(\Omega_{R_{\mathbb{X}}/K}^i) \mid i = 0, \dots, n\} \leq \max\{r_{\mathbb{X}} + n, \text{ri}(\Omega_{R_{\mathbb{X}}/K}^1) + n - 1\}.$$

Moreover, if we set  $\varrho_m = \max\{r_{\mathbb{X}} + m, \text{ri}(\Omega_{R_{\mathbb{X}}/K}^1) + m - 1\}$  for  $m \geq 1$ , then we get the upper bound for the regularity index of  $\Omega_{R_{\mathbb{X}}/K}^m$  as  $\text{ri}(\Omega_{R_{\mathbb{X}}/K}^m) \leq \min\{\varrho_n, \varrho_m\}$ .

## 6. BOUNDS FOR $\text{ri}(\Omega_{R_{\mathbb{W}}/K}^m)$ FOR A FAT POINT SCHEME $\mathbb{W}$

Let  $s \geq 1$ , and let  $\mathbb{X} = \{P_1, \dots, P_s\}$  be a set of  $s$  distinct  $K$ -rational points in  $\mathbb{P}^n$ . For  $i = 1, \dots, s$ , we let  $\wp_i$  be the associated prime ideal of  $P_i$  in  $S$ .

**Definition 6.1.** Given a sequence of positive integers  $m_1, \dots, m_s$ , the intersection  $I_{\mathbb{W}} := \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}$  is a saturated homogeneous ideal in  $S$  and is therefore the vanishing ideal of a 0-dimensional subscheme  $\mathbb{W}$  of  $\mathbb{P}^n$ .

- (a) The scheme  $\mathbb{W}$ , denoted by  $\mathbb{W} = m_1 P_1 + \dots + m_s P_s$ , is called a **fat point scheme** in  $\mathbb{P}^n$ . The homogeneous vanishing ideal of  $\mathbb{W}$  is  $I_{\mathbb{W}}$ . The number  $m_j$  is called the **multiplicity** of the point  $P_j$  for  $j = 1, \dots, s$ .
- (b) If  $m_1 = \dots = m_s = \nu$ , we denote  $\mathbb{W}$  also by  $\nu \mathbb{X}$  and call it an **equimultiple fat point scheme**.
- (c) For  $i \geq 1$ , the fat point scheme  $\mathbb{W}^{(i)} = (m_1 + i)P_1 + \dots + (m_s + i)P_s$  is called the  **$i$ -th fattening** of  $\mathbb{W}$ . We simply say the fattening of  $\mathbb{W}$  instead of the first fattening of  $\mathbb{W}$ .

The regularity index of the module of Kähler differential  $m$ -forms for fat point schemes can be bounded as follows.

**Proposition 6.2.** Let  $\mathbb{W} = m_1 P_1 + \dots + m_s P_s$  be a fat point scheme in  $\mathbb{P}^n$ , and let  $\mathbb{V} = \mathbb{W}^{(1)}$  be the fattening of  $\mathbb{W}$ .

- (a) For  $1 \leq m \leq n + 1$ , we have

$$\text{ri}(\Omega_{R_{\mathbb{W}}/K}^m) \leq \min\{\max\{r_{\mathbb{W}} + m, r_{\mathbb{V}} + m - 1\}, \max\{r_{\mathbb{W}} + n, r_{\mathbb{V}} + n - 1\}\}.$$

- (b) If  $m_1 \leq \dots \leq m_s$  and if  $\text{Supp}(\mathbb{W}) = \{P_1, \dots, P_s\}$  is in general position, then we have

$$\begin{aligned} \text{ri}(\Omega_{R_{\mathbb{W}}/K}^m) \leq \min\{ & \max\{m_s + m_{s-1} + m, \lfloor \frac{\sum_{j=1}^s m_j + s + n - 2}{n} \rfloor + m - 1\}, \\ & \max\{m_s + m_{s-1} + n, \lfloor \frac{\sum_{j=1}^s m_j + s + n - 2}{n} \rfloor + n - 1\} \} \\ & \text{for } 1 \leq m \leq n + 1. \end{aligned}$$

*Proof.* Claim (a) follows from Remark 5.4 and [KLL, Corollary 1.9(iii)]. Moreover, if  $\text{Supp}(\mathbb{W})$  is in general position, then [CTV, Theorem 6] implies that

$$\begin{aligned} \max\{r_{\mathbb{W}} + m, r_{\mathbb{V}} + m - 1\} & \leq \max\left\{m_s + m_{s-1} + m - 1, \left\lfloor \frac{\sum_{j=1}^s m_j + n - 2}{n} \right\rfloor + m, \right. \\ & \quad \left. m_s + m_{s-1} + m, \left\lfloor \frac{\sum_{j=1}^s m_j + s + n - 2}{n} \right\rfloor + m - 1 \right\} \\ & \leq \max\left\{m_s + m_{s-1} + m, \left\lfloor \frac{\sum_{j=1}^s m_j + s + n - 2}{n} \right\rfloor + m - 1 \right\}. \end{aligned}$$

Thus claim (b) follows from (a).  $\square$

The following example shows that the upper bounds for the regularity index of  $\Omega_{R_{\mathbb{W}}/K}^m$  given in Proposition 6.2 are sharp.

**Example 6.3.** Let  $K = \mathbb{Q}$ , and let  $\mathbb{W}$  be the fat point scheme

$$\mathbb{W} = P_1 + 2P_2 + P_3 + P_4 + 2P_5 + 2P_6 + 2P_7 + P_8 \subseteq \mathbb{P}^3$$

where  $P_1 = (1 : 9 : 0 : 0)$ ,  $P_2 = (1 : 6 : 0 : 1)$ ,  $P_3 = (1 : 2 : 3 : 3)$ ,  $P_4 = (1 : 9 : 3 : 5)$ ,  $P_5 = (1 : 3 : 0 : 4)$ ,  $P_6 = (1 : 0 : 1 : 3)$ ,  $P_7 = (1 : 0 : 2 : 0)$ , and  $P_8 = (1 : 3 : 0 : 10)$ . Let  $\mathbb{V}$  be the fat point scheme  $\mathbb{V} = 2P_1 + 3P_2 + 2P_3 + 2P_4 + 3P_5 + 3P_6 + 3P_7 + 2P_8$ . We have  $r_{\mathbb{W}} = 3$  and  $r_{\mathbb{V}} = 5$ , and so  $\max\{r_{\mathbb{W}} + m, r_{\mathbb{V}} + m - 1\} = m + 4$  for  $m = 1, \dots, 4$ . In this case the regularity index of  $\Omega_{R_{\mathbb{W}}/K}^m$  is  $m + 4$  for  $m = 1, \dots, 3$  and  $\text{ri}(\Omega_{R_{\mathbb{W}}/K}^4) = 7$ . Thus the bound for the regularity index in Proposition 6.2(a) is sharp.

Next let  $\mathbb{Y}$  be the scheme  $\mathbb{Y} = P_4 + P_5 + P_6 + P_7 + P_8$  in  $\mathbb{P}^3$ . Then  $\mathbb{Y}$  is in general position. For  $m = 1, 2, 3$ , the regularity index of  $\Omega_{R_{2\mathbb{Y}}/K}^m$  is  $4 + m$ . Thus, for  $m = 1, 2, 3$ , we have

$$\text{ri}(\Omega_{R_{2\mathbb{Y}}/K}^m) = 4 + m = \max \left\{ 2 + 2 + m, \left\lfloor \frac{\sum_{i=1}^5 2+5+3-2}{3} \right\rfloor + m - 1 \right\}.$$

In addition, for  $m = 4$ , we have

$$\text{ri}(\Omega_{R_{2\mathbb{Y}}/K}^4) = 7 = \max \left\{ 2 + 2 + 3, \left\lfloor \frac{\sum_{i=1}^5 2+5+3-2}{3} \right\rfloor + 3 - 1 \right\},$$

and hence the bound in Proposition 6.2(b) is also sharp.

## 7. BOUNDS FOR THE HILBERT POLYNOMIAL OF $\Omega_{R_{\mathbb{W}}/K}^m$ FOR A FAT POINT SCHEME $\mathbb{W}$

First we determine the Hilbert polynomial of the module of Kähler differential  $m$ -forms for a set of  $s$  distinct  $K$ -rational points  $\mathbb{X} = \{P_1, \dots, P_s\}$  in  $\mathbb{P}^n$ . Notice that all points of  $\mathbb{X}$  are assumed to lie outside the hyperplane  $\mathcal{Z}^+(X_0)$ , so we may write  $P_j = (1 : p_{j1} : \dots : p_{jn})$  with  $p_{j1}, \dots, p_{jn} \in K$  for  $j = 1, \dots, s$ . Furthermore, for every element  $f \in R_{\mathbb{X}}$  and  $j \in \{1, \dots, s\}$ , we also write  $f(P_j) = F(P_j)$ , where  $F$  is any representative of  $f$  in  $S$ .

Recall that an element  $f_j \in (R_{\mathbb{X}})_{r_{\mathbb{X}}}$  is the **normal separator** of  $\mathbb{X} \setminus \{P_j\}$  in  $\mathbb{X}$  if  $f_j(P_j) = 1$  and  $f_j(P_k) = 0$  for  $k \neq j$ . The set  $\{x_0^{i-r_{\mathbb{X}}} f_1, \dots, x_0^{i-r_{\mathbb{X}}} f_s\}$  is a  $K$ -basis of  $(R_{\mathbb{X}})_i$  for all  $i \geq r_{\mathbb{X}}$ . For more details about separators of a 0-dimensional scheme in  $\mathbb{P}^n$  see [GKR, GMT, Kre1, Kre2].

The following proposition gives a description of the Hilbert polynomial of the module of Kähler differential  $m$ -forms  $\Omega_{R_{\mathbb{X}}/K}^m$  for every  $1 \leq m \leq n + 1$ .

**Proposition 7.1.** *Let  $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$  be a set of  $s$  distinct  $K$ -rational points, and let  $1 \leq m \leq n + 1$ . We have*

$$\text{HP}_{\Omega_{R_{\mathbb{X}}/K}^m}(z) = \begin{cases} \deg(\mathbb{X}) & \text{if } m = 1, \\ 0 & \text{if } m \geq 2. \end{cases}$$

*In particular, the regularity index of  $\Omega_{R_{\mathbb{X}}/K}^m$  satisfies  $\text{ri}(\Omega_{R_{\mathbb{X}}/K}^m) \leq 2r_{\mathbb{X}} + m$ .*

*Proof.* For  $m = 1$ , we have  $\text{HP}_{\Omega_{R_{\mathbb{X}}/K}^1}(z) = \deg(\mathbb{X})$  and  $\text{ri}(\Omega_{R_{\mathbb{X}}/K}^1) \leq 2r_{\mathbb{X}} + 1$  (see [DK, Proposition 3.5]). Assume that  $m \geq 2$ . We see that  $\Omega_{R_{\mathbb{X}}/K}^m$  is a graded  $R_{\mathbb{X}}$ -module generated by the set of  $\binom{n+1}{m}$  elements

$$\{dx_{i_1} \wedge \cdots \wedge dx_{i_m} \mid 0 \leq i_1 < \cdots < i_m \leq n\}.$$

For  $j \in \{1, \dots, s\}$ , let  $f_j$  be the normal separator of  $\mathbb{X} \setminus \{P_j\}$  in  $\mathbb{X}$ . Since the set  $\{x_0^{i-r_{\mathbb{X}}}f_1, \dots, x_0^{i-r_{\mathbb{X}}}f_s\}$  is a  $K$ -basis of the  $K$ -vector space  $(R_{\mathbb{X}})_i$  for  $i \geq r_{\mathbb{X}}$ , the set

$$\{x_0^{k-r_{\mathbb{X}}-m}f_j dx_{i_1} \wedge \cdots \wedge dx_{i_m} \mid 0 \leq i_1 < \cdots < i_m \leq n, 1 \leq j \leq s\}$$

is a system of generators of the  $K$ -vector space  $(\Omega_{R_{\mathbb{X}}/K}^m)_k$  for all  $k \geq r_{\mathbb{X}} + m$ . Note that  $f_j^2 = f_j(P_j)x_0^{r_{\mathbb{X}}}f_j = x_0^{r_{\mathbb{X}}}f_j$  and  $x_i f_j = p_{ji}x_0 f_j$  (see, e.g., [GKR, Proposition 1.13]). Therefore we get

$$\begin{aligned} x_0^{r_{\mathbb{X}}}f_j dx_{i_1} \wedge \cdots \wedge dx_{i_m} &= f_j^2 dx_{i_1} \wedge \cdots \wedge dx_{i_m} \\ &= (d(f_j^2 x_{i_1}) - x_{i_1} df_j^2) \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m} \\ &= (d(p_{ji_1}x_0 f_j^2) - x_{i_1} df_j^2) \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m} \\ &= ((p_{ji_1}x_0 - x_{i_1})df_j^2 + p_{ji_1}f_j^2 dx_0) \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m} \\ &= (2(p_{ji_1}x_0 - x_{i_1})f_j df_j + p_{ji_1}f_j^2 dx_0) \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m} \\ &= p_{ji_1}f_j^2 dx_0 \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m} \\ &= p_{ji_1}x_0^{r_{\mathbb{X}}}f_j dx_0 \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}. \end{aligned}$$

Since  $m \geq 2$ , we may use the same method as above to get the equality

$$x_0^{r_{\mathbb{X}}}f_j dx_{i_1} \wedge \cdots \wedge dx_{i_m} = p_{ji_1}p_{ji_2}x_0^{r_{\mathbb{X}}}f_j dx_0 \wedge dx_{i_3} \wedge \cdots \wedge dx_{i_m}.$$

This implies  $x_0^{r_{\mathbb{X}}}f_j dx_{i_1} \wedge \cdots \wedge dx_{i_m} = 0$  for  $j = 1, \dots, s$  and  $\{i_1, \dots, i_m\} \subseteq \{0, \dots, n\}$ . Thus we obtain  $(\Omega_{R_{\mathbb{X}}/K}^m)_k = \langle 0 \rangle$  for all  $k \geq 2r_{\mathbb{X}} + m$ , and the claim follows.  $\square$

By combining Corollary 2.5 and Proposition 7.1, we obtain upper bounds for the regularity indices of modules of the Kähler differential  $m$ -forms  $\Omega_{R_{\mathbb{X}}/K}^m$  as follows.

**Corollary 7.2.** *In the setting of Proposition 7.1, we have*

$$\text{ri}(\Omega_{R_{\mathbb{X}}/K}^m) \leq \min\{2r_{\mathbb{X}} + m, 2r_{\mathbb{X}} + n\}.$$

The preceding bounds for the regularity indices of  $\Omega_{R_{\mathbb{X}}/K}^m$  are sharp, as our next example shows.

**Example 7.3.** Let  $K = \mathbb{Q}$ , and let  $\mathbb{X} \subseteq \mathbb{P}^3$  be the set of four  $K$ -rational points  $\mathbb{X} = \{P_1, P_2, P_3, P_4\}$ , where  $P_1 = (1 : 9 : 0 : 0)$ ,  $P_2 = (1 : 6 : 0 : 1)$ ,  $P_3 = (1 : 2 : 3 : 3)$ , and  $P_4 = (1 : 9 : 3 : 5)$ . It is clear that  $\text{HF}_{\mathbb{X}} : 1 \ 4 \ 4 \cdots$  and  $r_{\mathbb{X}} = 1$ . Moreover, we have

$$\begin{aligned} \text{HF}_{\Omega_{R_{\mathbb{X}}/K}^1} &: 0 \ 4 \ 10 \ 4 \ 4 \cdots, & \text{HF}_{\Omega_{R_{\mathbb{X}}/K}^2} &: 0 \ 0 \ 6 \ 4 \ 0 \ 0 \cdots, \\ \text{HF}_{\Omega_{R_{\mathbb{X}}/K}^3} &: 0 \ 0 \ 0 \ 4 \ 1 \ 0 \ 0 \cdots, & \text{HF}_{\Omega_{R_{\mathbb{X}}/K}^4} &: 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \cdots. \end{aligned}$$

It follows that  $\text{ri}(\Omega_{R_{\mathbb{X}}/K}^1) = 2r_{\mathbb{X}} + m = 2r_{\mathbb{X}} + 1 = 3$ ,  $\text{ri}(\Omega_{R_{\mathbb{X}}/K}^2) = 2r_{\mathbb{X}} + m = 2r_{\mathbb{X}} + 2 = 4$ , and  $\text{ri}(\Omega_{R_{\mathbb{X}}/K}^3) = \text{ri}(\Omega_{R_{\mathbb{X}}/K}^4) = \min\{2r_{\mathbb{X}} + m, 2r_{\mathbb{X}} + n\} = 2r_{\mathbb{X}} + n = 5$ . Hence we obtain the equality  $\text{ri}(\Omega_{R_{\mathbb{X}}/K}^m) = \min\{2r_{\mathbb{X}} + m, 2r_{\mathbb{X}} + n\}$  for  $m = 1, \dots, 4$ . Consequently, the bounds in Corollary 7.2 are sharp.

Now we give bounds for the Hilbert polynomial of the module of Kähler differential  $m$ -forms for a non-reduced fat point scheme.

**Proposition 7.4.** *Let  $\mathbb{W} = m_1 P_1 + \cdots + m_s P_s$  be a fat point scheme in  $\mathbb{P}^n$  such that  $m_i \geq 2$  for some  $i \in \{1, \dots, s\}$ , and let  $1 \leq m \leq n+1$ . The Hilbert polynomial of  $\Omega_{R_{\mathbb{W}}/K}^m$  is a constant polynomial which is bounded by*

$$\sum_{i=1}^s \binom{n+1}{m} \binom{m_i + n - 2}{n} \leq \text{HP}_{\Omega_{R_{\mathbb{W}}/K}^m}(z) \leq \sum_{i=1}^s \binom{n+1}{m} \binom{m_i + n - 1}{n}.$$

*Proof.* Let  $\mathbb{Y}$  be the subscheme  $\mathbb{Y} = (m_1 - 1)P_1 + \cdots + (m_s - 1)P_s$  of  $\mathbb{W}$ . Since we have  $dI_{\mathbb{W}} \subseteq I_{\mathbb{Y}}\Omega_{S/K}^1$ , this implies  $dI_{\mathbb{W}}\Omega_{S/K}^{m-1} \subseteq I_{\mathbb{Y}}\Omega_{S/K}^m$ . Obviously, we have the inclusion  $I_{\mathbb{W}} \subseteq I_{\mathbb{Y}}$ , and so  $I_{\mathbb{W}}\Omega_{S/K}^m \subseteq I_{\mathbb{Y}}\Omega_{S/K}^m$ . From this we deduce

$$I_{\mathbb{W}}\Omega_{S/K}^m + dI_{\mathbb{W}}\Omega_{S/K}^{m-1} \subseteq I_{\mathbb{Y}}\Omega_{S/K}^m.$$

By Proposition 2.2, the Hilbert function of  $\Omega_{R_{\mathbb{W}}/K}^m$  satisfies

$$\text{HF}_{\Omega_{R_{\mathbb{W}}/K}^m}(i) = \text{HF}_{\Omega_{S/K}^m / (I_{\mathbb{W}}\Omega_{S/K}^m + dI_{\mathbb{W}}\Omega_{S/K}^{m-1})}(i) \geq \text{HF}_{\Omega_{S/K}^m / I_{\mathbb{Y}}\Omega_{S/K}^m}(i)$$

for all  $i \in \mathbb{Z}$ . Also, we see that  $\text{HP}_{\Omega_{S/K}^m / I_{\mathbb{Y}}\Omega_{S/K}^m}(z) = \sum_{i=1}^s \binom{n+1}{m} \binom{m_i + n - 2}{n} > 0$  since  $m_i \geq 2$  for some  $i \in \{1, \dots, s\}$ . Hence we get the stated lower bound for the Hilbert polynomial of  $\Omega_{R_{\mathbb{W}}/K}^m$ . In particular, we have  $\text{HP}_{\Omega_{R_{\mathbb{W}}/K}^m}(z) > 0$ .

Furthermore, Proposition 4.2 shows that  $\text{HP}_{\Omega_{R_{\mathbb{W}}/K}^m}(z)$  is a constant polynomial. Now we find an upper bound for  $\text{HP}_{\Omega_{R_{\mathbb{W}}/K}^m}(z)$ . Clearly, the  $R_{\mathbb{W}}$ -module  $\Omega_{R_{\mathbb{W}}/K}^m$  is generated by the set  $\{dx_{i_1} \wedge \cdots \wedge dx_{i_m} \mid 0 \leq i_1 < \cdots < i_m \leq n\}$  consisting of  $\binom{n+1}{m}$  elements. This implies  $\text{HF}_{\Omega_{R_{\mathbb{W}}/K}^m}(i) \leq \binom{n+1}{m} \text{HF}_{\mathbb{W}}(i - m)$  for all  $i \geq 0$ . Hence we get  $\text{HP}_{\Omega_{R_{\mathbb{W}}/K}^m}(z) \leq \binom{n+1}{m} \sum_{i=1}^s \binom{m_i + n - 1}{n}$ , which completes the proof.  $\square$

Our next corollary is an immediate consequence of Propositions 7.1 and 7.4.

**Corollary 7.5.** *Let  $\mathbb{W} = m_1 P_1 + \cdots + m_s P_s$  be a fat point scheme in  $\mathbb{P}^n$ , and let  $m_{\max} = \max\{m_1, \dots, m_s\}$ . The following conditions are equivalent.*

- (a) *The scheme  $\mathbb{W}$  is not reduced, i.e.,  $m_{\max} > 1$ .*
- (b) *There exists  $m \in \{2, \dots, n+1\}$  such that  $\text{HP}_{\Omega_{R_{\mathbb{W}}/K}^m}(z) > 0$ .*
- (c)  $\text{HP}_{\Omega_{R_{\mathbb{W}}/K}^{n+1}}(z) > 0$ .

## 8. THE HILBERT POLYNOMIAL OF $\Omega_{R_{\mathbb{W}}/K}^{n+1}$ FOR A FAT POINT SCHEME $\mathbb{W}$

Given a fat point scheme  $\mathbb{W} = m_1 P_1 + \cdots + m_s P_s$  in  $\mathbb{P}^n$ , the Hilbert function of the  $R_{\mathbb{W}}$ -module  $\Omega_{R_{\mathbb{W}}/K}^1$  satisfies  $\text{HF}_{\Omega_{R_{\mathbb{W}}/K}^1}(i) = (n+1) \text{HF}_{\mathbb{W}}(i-1) + \text{HF}_{\mathbb{W}}(i) - \text{HF}_{\mathbb{V}}(i)$  for all  $i \in \mathbb{Z}$ , where  $\mathbb{V}$  is the fattening of  $\mathbb{W}$  (see [KLL, Corollary 1.9(i)]). Naturally, we still want to give a formula for the Hilbert function of the module  $\Omega_{R_{\mathbb{W}}/K}^m$  for  $m \geq 2$ . In fact, we can calculate the Hilbert function of  $\Omega_{R_{\mathbb{W}}/K}^{n+1}$  in the following special case.

**Proposition 8.1.** *Let  $\mathbb{W} = m_1 P_1 + \dots + m_s P_s$  be a fat point scheme in  $\mathbb{P}^n$  supported at a set of points  $\mathbb{X} = \{P_1, \dots, P_s\}$ , and let  $\mathbb{Y}$  be the subscheme  $\mathbb{Y} = (m_1 - 1)P_1 + \dots + (m_s - 1)P_s$  of  $\mathbb{W}$ . Suppose that  $\mathbb{X}$  is contained in a hyperplane. Then we have*

$$\Omega_{R_{\mathbb{W}}/K}^{n+1} \cong R_{\mathbb{Y}}(-n-1).$$

*In particular, we have  $\mathrm{HF}_{\Omega_{R_{\mathbb{W}}/K}^{n+1}}(i) = \mathrm{HF}_{\mathbb{Y}}(i - n - 1)$  for all  $i \in \mathbb{Z}$ .*

*Proof.* Assume that  $\mathbb{X} \subseteq \mathcal{Z}^+(H)$ , where  $0 \neq H = \sum_{i=0}^n a_i X_i \in S$  and  $a_0, \dots, a_n \in K$ . By letting  $I = \langle \frac{\partial F}{\partial X_i} \mid F \in I_{\mathbb{W}}, 0 \leq i \leq n \rangle$  and by Corollary 2.3, we have

$$\Omega_{R_{\mathbb{W}}/K}^{n+1} \cong (S/I)(-n-1).$$

Write  $I_{\mathbb{W}} = \wp_1^{m_1} \cap \dots \cap \wp_s^{m_s}$  where  $\wp_j$  is the associated prime ideal of  $P_j$  in  $S$ , and let  $F \in I_{\mathbb{W}} \setminus \{0\}$ . Clearly,  $\frac{\partial F}{\partial X_i} \in \wp_j^{m_j-1}$  for  $i = 0, \dots, n$  and  $j = 1, \dots, s$ , and so  $\frac{\partial F}{\partial X_i} \in I_{\mathbb{Y}}$  for  $i = 0, \dots, n$ . Consequently, we get  $I \subseteq I_{\mathbb{Y}}$ .

Now we prove  $I \supseteq I_{\mathbb{Y}}$ . Suppose for a contradiction that there exists a homogeneous polynomial  $G$  such that  $G \in I_{\mathbb{Y}} \setminus I$ . Then we have  $HG \in I_{\mathbb{W}}$ . Since  $H \neq 0$ , we may assume that  $a_i \neq 0$  for some  $i \in \{0, \dots, n\}$ . We have  $\frac{\partial(HG)}{\partial X_i} = a_i G + H \frac{\partial G}{\partial X_i} \in I$ . Since  $G \notin I$ , we deduce  $G_1 := H \frac{\partial G}{\partial X_i} \in I_{\mathbb{Y}} \setminus I$ . Furthermore, we continue to have  $HG_1 \in I_{\mathbb{W}}$ , and so

$$\frac{\partial(HG_1)}{\partial X_i} = \frac{\partial(H^2 \frac{\partial G}{\partial X_i})}{\partial X_i} = H^2 \frac{\partial^2 G}{\partial X_i^2} + 2a_i H \frac{\partial G}{\partial X_i} = H^2 \frac{\partial^2 G}{\partial X_i^2} + 2a_i G_1 \in I.$$

This implies that  $G_2 := H^2 \frac{\partial^2 G}{\partial X_i^2} \in I_{\mathbb{Y}} \setminus I$ . Repeating this process, we eventually get  $H^{\deg(G)} \in I_{\mathbb{Y}} \setminus I$ . On the other hand, since  $G \in I_{\mathbb{Y}}$ , it follows that  $\deg(G) \geq \max\{m_1 - 1, \dots, m_s - 1\}$ . Thus we have  $H^{\deg(G)+1} \in I_{\mathbb{W}}$ , and consequently  $H^{\deg(G)} = \frac{1}{a_i(\deg(G)+1)} \frac{\partial H^{\deg(G)+1}}{\partial X_i} \in I$ , a contradiction. Therefore we get  $I_{\mathbb{Y}} = I$ , and hence  $\Omega_{R_{\mathbb{W}}/K}^{n+1} \cong (S/I_{\mathbb{Y}})(-n-1)$ , as desired.  $\square$

Let us apply the preceding proposition to a concrete case.

**Example 8.2.** Let  $K = \mathbb{Q}$ , and let  $\mathbb{W}$  and  $\mathbb{Y}$  be the fat point schemes  $\mathbb{W} = 2P_1 + 3P_2 + 4P_3 + 2P_4 + P_5 + 7P_6 + 5P_7$  and  $\mathbb{Y} = P_1 + 2P_2 + 3P_3 + P_4 + 6P_6 + 4P_7$  in  $\mathbb{P}^5$ , where  $P_1 = (1 : 1 : 1 : 1 : 1 : \frac{15}{6})$ ,  $P_2 = (1 : 2 : 1 : 1 : 1 : \frac{17}{6})$ ,  $P_3 = (1 : 1 : 2 : 1 : 1 : \frac{18}{6})$ ,  $P_4 = (1 : 2 : 3 : 4 : 5 : \frac{55}{6})$ ,  $P_5 = (1 : 2 : 2 : 1 : 1 : \frac{20}{6})$ ,  $P_6 = (1 : 3 : 2 : 1 : 1 : \frac{22}{6})$ , and where  $P_7 = (1 : 0 : 0 : 1 : 1 : \frac{10}{6})$ . Then  $\mathbb{X} = \{P_1, \dots, P_7\}$  is contained in the hyperplane  $\mathcal{Z}^+(X_0 - 4X_3 + 3X_4)$ . Thus Proposition 8.1 yields  $\Omega_{R_{\mathbb{W}}/K}^6 \cong R_{\mathbb{Y}}(-6)$ . Hence the Hilbert function of  $\Omega_{R_{\mathbb{W}}/K}^6$  is given by

$$\mathrm{HF}_{\Omega_{R_{\mathbb{W}}/K}^6} : 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 6 \ 21 \ 56 \ 126 \ 252 \ 306 \ 329 \ 336 \ 337 \ 337 \ \dots$$

Although no formula for the Hilbert function of the module of Kähler differential  $(n+1)$ -forms of an equimultiple fat point scheme is known, the following theorem provides a formula for its Hilbert polynomial.

**Theorem 8.3.** *Let  $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$  be a set of  $s$  distinct  $K$ -rational points, and let  $\nu \geq 1$ . Then we have  $\mathrm{HP}_{\Omega_{R_{(\nu+1)\mathbb{X}}/K}^{n+1}}(z) = \mathrm{HP}_{\nu\mathbb{X}}(z)$ .*

*Proof.* Let  $I = \langle \frac{\partial F}{\partial X_i} \mid F \in I_{\mathbb{X}}, 0 \leq i \leq n \rangle$ . By Corollary 2.3, we have  $\Omega_{R_{\mathbb{X}}/K}^{n+1} \cong (S/I)(-n-1)$ . So, Proposition 7.1 implies  $\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^{n+1}}(i) = \mathrm{HF}_{S/I}(i - n - 1) = 0$  for



$i \gg 0$ . Let  $\mathfrak{M}$  denote the homogeneous maximal ideal of  $S$ . There exists a number  $t_1 \in \mathbb{N}$  such that  $I_{t_1+i} = \mathfrak{M}_{t_1+i}$  for all  $i \in \mathbb{N}$ . Moreover, by [HC, 4.2], there is  $t_2 \in \mathbb{N}$  such that  $(I_{\nu\mathbb{X}})_{t_2+i} = (I_{\mathbb{X}}^\nu)_{t_2+i}$  for all  $i \in \mathbb{N}$ . Let  $t = \max\{t_1, t_2, r_{\nu\mathbb{X}} + 1\}$ , let  $r = \binom{n+t}{n} - s$ , let  $\{F_1, \dots, F_r\}$  be a  $K$ -basis of the  $K$ -vector space  $(I_{\mathbb{X}})_t$ , and let

$$J = \left\langle \frac{\partial F}{\partial X_i} \mid F \in I_{(\nu+1)\mathbb{X}}, 0 \leq i \leq n \right\rangle.$$

Clearly, we have  $I_{(\nu+1)\mathbb{X}} \subseteq J \subseteq I_{\nu\mathbb{X}}$ . Since  $\Omega_{R_{(\nu+1)\mathbb{X}}/K}^{n+1} \cong (S/J)(-n-1)$ , and since  $I_{\nu\mathbb{X}}$  is generated by elements of degrees  $\leq t$  (cf. [GM, Proposition 1.1]), it suffices to show that  $J_i = (I_{\nu\mathbb{X}})_i$  for some  $i \geq t$ .

We observe that

$$\begin{aligned} I_t &= \left\langle \frac{\partial F}{\partial X_i} \mid F \in I_{\mathbb{X}}, 0 \leq i \leq n \right\rangle + I_{\mathbb{X}} \Big|_t \\ &= \left\langle \left\{ \frac{\partial F}{\partial X_i} \mid F \in (I_{\mathbb{X}})_{t+1}, 0 \leq i \leq n \right\} \cup \left\{ G \frac{\partial H}{\partial X_i} \mid G \in (I_{\mathbb{X}})_k, H \in S_{t+1-k}, 0 \leq i \leq n \right\} \right\rangle_K \\ &= \left\langle \frac{\partial F}{\partial X_i} \mid F \in (I_{\mathbb{X}})_{t+1}, 0 \leq i \leq n \right\rangle_K + (I_{\mathbb{X}})_t \\ &= \left\langle \frac{\partial(X_j G)}{\partial X_i} \mid G \in (I_{\mathbb{X}})_t, 0 \leq i, j \leq n \right\rangle_K + (I_{\mathbb{X}})_t \\ &= \left\langle \frac{\partial F_j}{\partial X_i} \mid 0 \leq i \leq n, 1 \leq j \leq r \right\rangle_K \mathfrak{M}_1. \end{aligned}$$

Here the last two equalities follow from Euler's relation and the fact that  $(I_{\mathbb{X}})_{t+1} = (I_{\mathbb{X}})_t \mathfrak{M}_1 = \langle F_j \mid 1 \leq j \leq r \rangle_K \mathfrak{M}_1$ . (Note that  $I_{\mathbb{X}}$  can be generated in degrees  $\leq t$ .) Therefore we get equalities

$$\begin{aligned} (I_{\nu\mathbb{X}})_{(\nu r n + \nu + 1)t} &= (I_{\nu\mathbb{X}})_{\nu t} \cdot \mathfrak{M}_{(\nu r n + 1)t} \\ &= (I_{\nu\mathbb{X}})_{\nu t} \cdot (\mathfrak{M}^{\nu r n + 1})_{(\nu r n + 1)t} \\ &= (I_{\mathbb{X}}^\nu)_{\nu t} \cdot (\mathfrak{M}^{\nu r n + 1})_{(\nu r n + 1)t} \\ &= \underbrace{(I_{\mathbb{X}})_t \cdots (I_{\mathbb{X}})_t}_{\nu \text{ times}} \cdot \underbrace{\mathfrak{M}_t \cdots \mathfrak{M}_t}_{\nu r n + 1 \text{ times}} \\ &= \underbrace{(I_{\mathbb{X}})_t \cdots (I_{\mathbb{X}})_t}_{\nu \text{ times}} \cdot \underbrace{I_t \cdots I_t}_{\nu r n + 1 \text{ times}} \\ &= \langle F_1, \dots, F_r \rangle_K^\nu \cdot \langle \partial F_j / \partial X_i \mid 0 \leq i \leq n, 1 \leq j \leq r \rangle_K \mathfrak{M}_1^{\nu r n + 1}. \end{aligned}$$

Thus we only need to prove the inclusion

$$\langle F_1, \dots, F_r \rangle_K^\nu \cdot \langle \partial F_j / \partial X_i \mid 0 \leq i \leq n, 1 \leq j \leq r \rangle_K \mathfrak{M}_1^{\nu r n + 1} \subseteq J.$$

To this end, we first prove that  $F_{j_1}^{\nu-k} F_{j_2} \cdots F_{j_{k+1}} \frac{\partial F_{j_1}}{\partial X_{i_1}} \cdots \frac{\partial F_{j_1}}{\partial X_{i_{k+1}}} \in J$  for all  $i_1, \dots, i_{k+1} \in \{0, \dots, n\}$  and  $j_1, \dots, j_{k+1} \in \{1, \dots, r\}$  and  $0 \leq k \leq \nu$ . We proceed by induction on  $k$ . If  $k = 0$ , for  $0 \leq i_1 \leq n$  and  $1 \leq j_1 \leq r$  we have

$$(\nu + 1) F_{j_1}^\nu \frac{\partial F_{j_1}}{\partial X_{i_1}} = \frac{\partial F_{j_1}^{\nu+1}}{\partial X_{i_1}} \in J.$$

If  $k = 1$ , for  $i_1, i_2 \in \{0, \dots, n\}$  and  $j_1, j_2 \in \{1, \dots, r\}$ , we get

$$\nu F_{j_1}^{\nu-1} F_{j_2} \frac{\partial F_{j_1}}{\partial X_{i_1}} \cdot \frac{\partial F_{j_1}}{\partial X_{i_2}} = \frac{\partial(F_{j_1}^\nu F_{j_2})}{\partial X_{i_1}} \cdot \frac{\partial F_{j_1}}{\partial X_{i_2}} - F_{j_1}^\nu \frac{\partial F_{j_1}}{\partial X_{i_2}} \cdot \frac{\partial F_{j_2}}{\partial X_{i_1}} \in J.$$

Now we assume that  $2 \leq k \leq \nu$  and  $F_{j_1}^{\nu-(k-1)} F_{j_2'} \cdots F_{j_k'} \frac{\partial F_{j_1}}{\partial X_{i_1'}} \cdots \frac{\partial F_{j_1}}{\partial X_{i_k'}} \in J$  for all  $i_1', \dots, i_k' \in \{0, \dots, n\}$  and  $j_1', \dots, j_k' \in \{1, \dots, r\}$ . Let  $i_1, \dots, i_{k+1} \in \{0, \dots, n\}$  and

$j_1, \dots, j_{k+1} \in \{1, \dots, r\}$ . We have

$$\begin{aligned} & (\nu - k + 1) F_{j_1}^{\nu-k} F_{j_2} \cdots F_{j_{k+1}} \frac{\partial F_{j_1}}{\partial X_{i_1}} \cdots \frac{\partial F_{j_1}}{\partial X_{i_{k+1}}} \\ &= \frac{\partial(F_{j_1}^{\nu-k+1} F_{j_2} \cdots F_{j_{k+1}})}{\partial X_{i_1}} \frac{\partial F_{j_1}}{\partial X_{i_2}} \cdots \frac{\partial F_{j_1}}{\partial X_{i_{k+1}}} - F_{j_1}^{\nu-k+1} \frac{\partial F_{j_2} \cdots F_{j_{k+1}}}{\partial X_{i_1}} \frac{\partial F_{j_1}}{\partial X_{i_2}} \cdots \frac{\partial F_{j_1}}{\partial X_{i_{k+1}}} \\ &= \frac{\partial(F_{j_1}^{\nu-k+1} F_{j_2} \cdots F_{j_{k+1}})}{\partial X_{i_1}} \frac{\partial F_{j_1}}{\partial X_{i_2}} \cdots \frac{\partial F_{j_1}}{\partial X_{i_{k+1}}} - \sum_{l=2}^{k+1} F_{j_1}^{\nu-(k-1)} \frac{F_{j_2} \cdots F_{j_{k+1}}}{F_{j_l}} \frac{\partial F_{j_1}}{\partial X_{i_1}} \frac{\partial F_{j_1}}{\partial X_{i_2}} \cdots \frac{\partial F_{j_1}}{\partial X_{i_{k+1}}}. \end{aligned}$$

By the inductive hypothesis, we have  $F_{j_1}^{\nu-(k-1)} \frac{F_{j_2} \cdots F_{j_{k+1}}}{F_{j_l}} \frac{\partial F_{j_1}}{\partial X_{i_2}} \cdots \frac{\partial F_{j_1}}{\partial X_{i_{k+1}}} \in J$  for all  $l = 2, \dots, k+1$ . Hence we get  $F_{j_1}^{\nu-k} F_{j_2} \cdots F_{j_{k+1}} \frac{\partial F_{j_1}}{\partial X_{i_1}} \cdots \frac{\partial F_{j_1}}{\partial X_{i_{k+1}}} \in J$ .

Consequently, if  $k = \nu$ , then we have  $F_{j_1} F_{j_2} \cdots F_{j_\nu} \frac{\partial F_{j_1}}{\partial X_{i_1}} \cdots \frac{\partial F_{j_1}}{\partial X_{i_{\nu+1}}} \in J$  for all  $i_1, \dots, i_{\nu+1} \in \{0, \dots, n\}$  and  $j, j_1, \dots, j_\nu \in \{1, \dots, r\}$ . On the other hand, the  $K$ -vector space  $\langle \frac{\partial F_j}{\partial X_i} \mid 0 \leq i \leq n, 1 \leq j \leq r \rangle_K^{\nu r+1}$  has a system of generators consisting of elements of the form

$$\left(\frac{\partial F_1}{\partial X_0}\right)^{\alpha_{01}} \cdots \left(\frac{\partial F_1}{\partial X_n}\right)^{\alpha_{n1}} \cdots \left(\frac{\partial F_r}{\partial X_0}\right)^{\alpha_{0r}} \cdots \left(\frac{\partial F_r}{\partial X_n}\right)^{\alpha_{nr}}$$

where  $\alpha_{ij} \in \mathbb{N}$  satisfy  $\sum_{j=1}^r \sum_{i=0}^n \alpha_{ij} = \nu r + 1$ . Set  $\alpha_j := \sum_{i=0}^n \alpha_{ij}$  for  $j = 1, \dots, r$ . Since  $\sum_{j=1}^r \alpha_j = \nu r + 1$ , there is an index  $j \in \{1, \dots, r\}$  such that  $\alpha_j \geq \nu + 1$ . Also, we have  $\nu r n + 1 \geq \nu r + 1$ . It follows that any element of the  $K$ -vector space

$$\langle F_1, \dots, F_r \rangle_K^\nu \cdot \langle \partial F_j / \partial X_i \mid 0 \leq i \leq n, 1 \leq j \leq r \rangle_K^{\nu r n + 1}$$

is a sum of elements of the form  $F_{j_1} \cdots F_{j_\nu} \frac{\partial F_j}{\partial X_{i_1}} \cdots \frac{\partial F_j}{\partial X_{i_{\nu+1}}} G$ , where  $i_1, \dots, i_{\nu+1} \in \{0, \dots, n\}$ , where  $j, j_1, \dots, j_\nu \in \{1, \dots, r\}$ , and where  $G \in S$  is a homogeneous polynomial of degree  $\deg(G) = \nu(rn - 1)(t - 1)$ . Therefore we get

$$\langle F_1, \dots, F_r \rangle_K^\nu \cdot (\langle \partial F_j / \partial X_i \mid 0 \leq i \leq n, 1 \leq j \leq r \rangle_K \mathfrak{M}_1)^{\nu r n + 1} \subseteq J,$$

and this completes the proof.  $\square$

The following corollary follows immediately from Theorem 8.3.

**Corollary 8.4.** *Let  $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$  be a set of  $s$  distinct  $K$ -rational points, and let  $\nu \geq 1$ . Then we have  $\mathrm{HP}_{R_{(\nu+1)\mathbb{X}/K}}^{n+1}(z) = s \binom{\nu+n-1}{n}$ .*

In the last part of this section we study the module of Kähler differential 2-forms of fat point schemes  $\mathbb{W}$  in  $\mathbb{P}^n$ . Let us begin with the following sequence of graded  $R_{\mathbb{W}}$ -modules.

**Proposition 8.5.** *Let  $\mathbb{W} = m_1 P_1 + \cdots + m_s P_s$  be a fat point scheme in  $\mathbb{P}^n$ , and let  $\mathbb{W}^{(i)}$  be the  $i$ -th fattening of  $\mathbb{W}$  for  $i \geq 1$ . Then the sequence of graded  $R_{\mathbb{W}}$ -modules*

$$(C) \quad 0 \rightarrow I_{\mathbb{W}(1)}/I_{\mathbb{W}(2)} \xrightarrow{\alpha} I_{\mathbb{W}} \Omega_{S/K}^1 / I_{\mathbb{W}(1)} \Omega_{S/K}^1 \xrightarrow{\beta} \Omega_{S/K}^2 / I_{\mathbb{W}} \Omega_{S/K}^2 \xrightarrow{\gamma} \Omega_{R_{\mathbb{W}}/K}^2 \rightarrow 0$$

*is a complex, where  $\alpha(F + I_{\mathbb{W}(2)}) = dF + I_{\mathbb{W}(1)} \Omega_{S/K}^1$ , where  $\beta(GdX_i + I_{\mathbb{W}(1)} \Omega_{S/K}^1) = d(GdX_i) + I_{\mathbb{W}} \Omega_{S/K}^2$ , and where  $\gamma(H + I_{\mathbb{W}} \Omega_{S/K}^2) = H + (I_{\mathbb{W}} \Omega_{S/K}^2 + dI_{\mathbb{W}} \Omega_{S/K}^1)$ . Moreover, the following statements hold true.*

- (a) *The map  $\alpha$  is injective.*
- (b) *The map  $\gamma$  is surjective.*
- (c) *We have  $\mathrm{Im}(\beta) = \mathrm{Ker}(\gamma)$ .*

(d) For all  $i \geq 0$ , we have

$$\begin{aligned} \mathrm{HF}_{\Omega_{R_{\mathbb{W}}/K}^2}(i+2) &\geq \frac{n(n+1)}{2} \mathrm{HF}_{\mathbb{W}}(i) + \mathrm{HF}_{\mathbb{W}^{(2)}}(i+2) - \mathrm{HF}_{\mathbb{W}^{(1)}}(i+2) \\ &\quad - (n+1)(\mathrm{HF}_{\mathbb{W}^{(1)}}(i+1) - \mathrm{HF}_{\mathbb{W}}(i+1)). \end{aligned}$$

*Proof.* To prove (a), we note that the arguments in the proof of [KLL, Theorem 1.7] shown that the map  $\alpha$  is a homogeneous injection. Claims (b) and (c) are implied by the presentation

$$\Omega_{R_{\mathbb{W}}/K}^2 \cong \Omega_{S/K}^2 / (I_{\mathbb{W}}\Omega_{S/K}^2 + dI_{\mathbb{W}}\Omega_{S/K}^1).$$

Since  $d \circ d = 0$ , it follows that  $\beta \circ \alpha = 0$ . Therefore the sequence  $(\mathcal{C})$  is a complex. In addition, claim (d) follows from the fact that  $(\mathcal{C})$  is a complex and (c).  $\square$

Now we look at the special case of an equimultiple fat point scheme  $\mathbb{W} = \nu\mathbb{X}$  in  $\mathbb{P}^2$ . In this case, by taking the homogeneous components at a large degree  $i$  of the complex  $(\mathcal{C})$ , we have the following exact sequence of  $K$ -vector spaces.

**Proposition 8.6.** *Let  $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^2$  be a set of  $s$  distinct  $K$ -rational points, and let  $\nu \geq 1$ . For  $i \gg 0$ , the sequence of  $K$ -vector spaces*

$$\begin{aligned} 0 \longrightarrow (I_{(\nu+1)\mathbb{X}}/I_{(\nu+2)\mathbb{X}})_i &\xrightarrow{\alpha} (I_{\nu\mathbb{X}}\Omega_{S/K}^1/I_{(\nu+1)\mathbb{X}}\Omega_{S/K}^1)_i \\ &\xrightarrow{\beta} (\Omega_{S/K}^2/I_{\nu\mathbb{X}}\Omega_{S/K}^2)_i \xrightarrow{\gamma} (\Omega_{R_{\nu\mathbb{X}}/K}^2)_i \longrightarrow 0. \end{aligned}$$

is exact. Here the maps  $\alpha$ ,  $\beta$  and  $\gamma$  are defined as in Proposition 8.5.

*Proof.* By Proposition 8.5, we only need to prove  $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$ . Hence it suffices to show that the Hilbert polynomial of  $\Omega_{R_{\nu\mathbb{X}}/K}^2$  satisfies

$$\mathrm{HP}_{\Omega_{R_{\nu\mathbb{X}}/K}^2}(z) = \frac{(n+2)(n+1)}{2} \mathrm{HP}_{\nu\mathbb{X}}(z) + \mathrm{HP}_{(\nu+2)\mathbb{X}}(z) - (n+2) \mathrm{HP}_{(\nu+1)\mathbb{X}}(z).$$

In  $\mathbb{P}^2$ , Proposition 2.4 yields the exact sequence of graded  $R_{\nu\mathbb{X}}$ -modules

$$0 \longrightarrow \Omega_{R_{\nu\mathbb{X}}/K}^3 \longrightarrow \Omega_{R_{\nu\mathbb{X}}/K}^2 \longrightarrow \Omega_{R_{\nu\mathbb{X}}/K}^1 \longrightarrow \mathfrak{m}_{\nu\mathbb{X}} \longrightarrow 0.$$

Moreover, we have  $\mathrm{HP}_{\Omega_{R_{\nu\mathbb{X}}/K}^3}(z) = \mathrm{HP}_{(\nu-1)\mathbb{X}}(z)$  by Theorem 8.3. Hence, by applying [KLL, Corollary 1.9], we get

$$\begin{aligned} \mathrm{HP}_{\Omega_{R_{\nu\mathbb{X}}/K}^2}(z) &= \mathrm{HP}_{\Omega_{R_{\nu\mathbb{X}}/K}^1}(z) + \mathrm{HP}_{\Omega_{R_{\nu\mathbb{X}}/K}^3}(z) - \mathrm{HP}_{\nu\mathbb{X}}(z) \\ &= ((n+2) \mathrm{HP}_{\nu\mathbb{X}}(z) - \mathrm{HP}_{(\nu+1)\mathbb{X}}(z)) + \mathrm{HP}_{(\nu-1)\mathbb{X}}(z) - \mathrm{HP}_{\nu\mathbb{X}}(z) \\ &= 3s \binom{\nu+1}{2} - s \binom{\nu+2}{2} + s \binom{\nu}{2} \\ &= \frac{1}{2}s(3\nu^2 - \nu - 2) \\ &= 6s \binom{\nu+1}{2} + s \binom{\nu+3}{2} - 4s \binom{\nu+2}{2} \\ &= \frac{(n+2)(n+1)}{2} \mathrm{HP}_{\nu\mathbb{X}}(z) + \mathrm{HP}_{(\nu+2)\mathbb{X}}(z) - (n+2) \mathrm{HP}_{(\nu+1)\mathbb{X}}(z), \end{aligned}$$

and the claim follows.  $\square$

The following formula for the Hilbert polynomial of  $\Omega_{R_{\nu\mathbb{X}}/K}^2$  can be extracted from the proof of this proposition.

**Corollary 8.7.** *Let  $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^2$  be a set of  $s$  distinct  $K$ -rational points, and let  $\nu \geq 1$ . Then we have  $\mathrm{HP}_{\Omega_{R_{\nu\mathbb{X}}/K}^2}(z) = \frac{1}{2}(3\nu^2 - \nu - 2)s$ .*

In general, we do not have an explicit formula for the Hilbert polynomial of  $\Omega_{R_{\mathbb{W}}/K}^{n+1}$  for a non-reduced fat point scheme  $\mathbb{W}$  in  $\mathbb{P}^n$ . However, we propose the following conjecture.

**Conjecture 8.8.** Let  $\mathbb{W}$  be the fat point scheme  $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$  in  $\mathbb{P}^n$ , and let  $\mathbb{Y} = (m_1 - 1)P_1 + \cdots + (m_s - 1)P_s$ . Then we have  $\text{HP}_{\Omega_{R_{\mathbb{W}}/K}^{n+1}}(z) = \text{HP}_{\mathbb{Y}}(z)$ .

The above conjecture holds true for  $n = 1$  by Proposition 3.1, and for  $n \geq 2$  such that  $\mathbb{W}$  is an equimultiple fat point scheme in  $\mathbb{P}^n$  by Theorem 8.3. The following example provides a further instance in which the conjecture holds.

**Example 8.9.** Let  $\mathbb{X} = \{P_1, \dots, P_{10}\}$  be the set of 10 points on the twisted cubic curve in  $\mathbb{P}^3$  given by  $P_1 = (1 : 1 : 1 : 1)$ ,  $P_2 = (1 : -1 : 1 : -1)$ ,  $P_3 = (1 : 2 : 4 : 8)$ ,  $P_4 = (1 : -2 : 4 : -8)$ ,  $P_5 = (1 : 3 : 9 : 27)$ ,  $P_6 = (1 : -3 : 9 : -27)$ ,  $P_7 = (1 : 4 : 16 : 64)$ ,  $P_8 = (1 : -4 : 16 : -64)$ ,  $P_9 = (1 : -5 : 25 : -125)$ , and  $P_{10} = (1 : 6 : 36 : 216)$ . Let  $\mathbb{W} = P_1 + 2P_2 + 4P_3 + 3P_4 + 4P_5 + 2P_6 + 3P_7 + 7P_8 + 5P_9 + 6P_{10}$ , and let  $\mathbb{Y} = P_2 + 3P_3 + 2P_4 + 3P_5 + P_6 + 2P_7 + 6P_8 + 4P_9 + 5P_{10}$ . A calculation yields that the Hilbert polynomials of  $\Omega_{R_{\mathbb{W}}/K}^{n+1}$  and of  $R_{\mathbb{Y}}$  are the same and equal to 141.

## 9. FAT POINT SCHEMES SUPPORTED ON NON-SINGULAR CONICS

In Proposition 3.1, we described concretely the Hilbert function of the module of Kähler differential  $m$ -forms when  $\mathbb{W}$  is a fat point scheme in  $\mathbb{P}^1$ . This result leads us to the following question:

*Question 9.1.* Can one compute explicitly the Hilbert function of the bi-graded  $R_{\mathbb{W}}$ -algebra  $\Omega_{R_{\mathbb{W}}/K}$  for a fat point scheme  $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$  in  $\mathbb{P}^2$ ?

In this section we answer some parts of this question. In particular, we give concrete formulas for the Hilbert function of the bi-graded algebra  $\Omega_{R_{\mathbb{W}}/K}$  if  $\mathbb{W}$  is an equimultiple fat point scheme in  $\mathbb{P}^2$  whose support  $\mathbb{X}$  lies on a non-singular conic.

In what follows, we let  $\mathcal{C} = \mathcal{Z}^+(C)$  be a non-singular conic defined by a quadratic polynomial  $C \in S = K[X_0, X_1, X_2]$ , we let  $\mathbb{X} = \{P_1, \dots, P_s\}$  be a set of  $s$  distinct  $K$ -rational points on  $\mathcal{C}$ , and we let  $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$  be a fat point scheme in  $\mathbb{P}^2$  supported at  $\mathbb{X}$ . Suppose that  $0 \leq m_1 \leq \cdots \leq m_s$  and  $s \geq 4$ . Then it was shown in [Cat, Proposition 2.2] that the regularity index of  $\mathbb{W}$  is

$$r_{\mathbb{W}} = \max \left\{ m_s + m_{s-1} - 1, \left\lfloor \sum_{j=1}^s m_j / 2 \right\rfloor \right\}.$$

Moreover, the Hilbert function of  $\mathbb{W}$  can be effectively computed from the Hilbert function of a certain subscheme  $\mathbb{Y}$  of  $\mathbb{W}$  (see [Cat, Theorem 3.1]).

The Hilbert function of the module of Kähler differential 1-forms  $\Omega_{R_{\mathbb{W}}/K}^1$  satisfies the following conditions.

**Theorem 9.2.** *In the setting above, let  $\mu = \sum_{j=1}^s m_j + s$  and  $\varrho = m_s + m_{s-1}$ .*

(a) If  $\mu \geq 2\varrho + 4$  then

$$\mathrm{HF}_{\Omega_{R_{\mathbb{W}}/K}^1}(i) = \begin{cases} \sum_{j=1}^s \frac{(m_j+1)(3m_j-2)}{2} & \text{if } i \geq \lfloor \frac{\mu}{2} \rfloor, \\ 3 \sum_{j=1}^s \binom{m_j+1}{2} - 2i - 1 & \text{if } r_{\mathbb{W}} + 2 \leq i < \lfloor \frac{\mu}{2} \rfloor, \\ 4 \sum_{j=1}^s \binom{m_j+1}{2} - 2i - 1 - \mathrm{HF}_{\mathbb{W}}(i-2) & \text{if } i = r_{\mathbb{W}} + 1, \\ \mathrm{HF}_{\mathbb{W}}(i) + 3 \mathrm{HF}_{\mathbb{W}}(i-1) - \mathrm{HF}_{\mathbb{W}}(i-2) - 2i - 1 & \text{if } 0 \leq i \leq r_{\mathbb{W}}. \end{cases}$$

(b) If  $\mu \leq 2\varrho + 3$  and  $\mathbb{Y} := (m_1+1)P_1 + \cdots + (m_{s-2}+1)P_{s-2} + m_{s-1}P_{s-1} + m_sP_s$ , then

$$\mathrm{HF}_{\Omega_{R_{\mathbb{W}}/K}^1}(i) = \begin{cases} \sum_{j=1}^s \frac{(m_j+1)(3m_j-2)}{2} & \text{if } i \geq \varrho + 1, \\ 4 \sum_{j=1}^s \binom{m_j+1}{2} - i - 1 - \mathrm{HF}_{\mathbb{Y}}(i-1) & \text{if } r_{\mathbb{W}} + 1 \leq i \leq \varrho, \\ \mathrm{HF}_{\mathbb{W}}(i) + 3 \mathrm{HF}_{\mathbb{W}}(i-1) - \mathrm{HF}_{\mathbb{Y}}(i-1) - i - 1 & \text{if } 0 \leq i \leq r_{\mathbb{W}}. \end{cases}$$

*Proof.* Let  $\mathbb{V}$  be the fat point scheme  $\mathbb{V} = (m_1+1)P_1 + \cdots + (m_s+1)P_s$  containing  $\mathbb{W}$ . By [KLL, Corollary 1.9(i)], we have

$$\mathrm{HF}_{\Omega_{R_{\mathbb{W}}/K}^1}(i) = 3 \mathrm{HF}_{\mathbb{W}}(i-1) + \mathrm{HF}_{\mathbb{W}}(i) - \mathrm{HF}_{\mathbb{V}}(i)$$

for all  $i \in \mathbb{Z}$ . Also, we have  $r_{\mathbb{V}} = \max\{\varrho + 1, \lfloor \frac{\mu}{2} \rfloor\}$ .

(a) Consider the case  $\mu \geq 2\varrho + 4$ . In this case, we have  $r_{\mathbb{V}} = \lfloor \frac{\mu}{2} \rfloor$ . Since  $s \geq 4$ , we get the inequality

$$r_{\mathbb{W}} + 1 = \max\{\varrho - 1, \lfloor \frac{\mu-s}{2} \rfloor\} + 1 < \lfloor \frac{\mu}{2} \rfloor = r_{\mathbb{V}}.$$

So, [KLL, Corollary 1.9(iii)] yields  $\mathrm{ri}(\Omega_{R_{\mathbb{W}}/K}^1) \leq r_{\mathbb{V}} = \lfloor \frac{\mu}{2} \rfloor$ . Moreover, we see that

$$\begin{aligned} \mathrm{HF}_{\Omega_{R_{\mathbb{W}}/K}^1}(r_{\mathbb{V}} - 1) &= 4 \deg(\mathbb{W}) - \mathrm{HF}_{\mathbb{V}}(r_{\mathbb{V}} - 1) > 4 \deg(\mathbb{W}) - \mathrm{HF}_{\mathbb{V}}(r_{\mathbb{V}} + i) \\ &= 4 \deg(\mathbb{W}) - \deg(\mathbb{V}) = \mathrm{HF}_{\Omega_{R_{\mathbb{W}}/K}^1}(r_{\mathbb{V}} + i) \end{aligned}$$

for all  $i \geq 0$ , and hence  $\mathrm{ri}(\Omega_{R_{\mathbb{W}}/K}^1) = r_{\mathbb{V}}$ . Consequently, we can apply [Cat, Theorem 3.1] to work out the Hilbert function of  $\Omega_{R_{\mathbb{W}}/K}^1$  with respect to the value of the degree  $i$  as follows:

(i) For  $i \geq r_{\mathbb{V}} = \lfloor \frac{\mu}{2} \rfloor$ , the Hilbert function of  $\Omega_{R_{\mathbb{W}}/K}^1$  satisfies

$$\mathrm{HF}_{\Omega_{R_{\mathbb{W}}/K}^1}(i) = 4 \sum_{j=1}^s \binom{m_j+1}{2} - \binom{m_j+2}{2} = \sum_{j=1}^s \frac{(m_j+1)(3m_j-2)}{2}.$$

(ii) Let  $r_{\mathbb{W}} + 2 \leq i < \lfloor \frac{\mu}{2} \rfloor$ . Then we have  $\mathrm{HF}_{\mathbb{W}}(i-2) = \mathrm{HF}_{\mathbb{W}}(i-1) = \mathrm{HF}_{\mathbb{W}}(i) = \deg(\mathbb{W}) = \sum_{j=1}^s \binom{m_j+1}{2}$  and  $\mathrm{HF}_{\mathbb{V}}(i) = 2i + 1 + \mathrm{HF}_{\mathbb{W}}(i-2)$ . It follows that

$$\mathrm{HF}_{\Omega_{\mathbb{W}}/K}^1(i) = 4 \sum_{j=1}^s \binom{m_j+1}{2} - 2i - 1 - \mathrm{HF}_{\mathbb{W}}(i-2) = 3 \sum_{j=1}^s \binom{m_j+1}{2} - 2i - 1.$$

(iii) In the case  $i = r_{\mathbb{W}} + 1$ , we have  $\mathrm{HF}_{\mathbb{W}}(i-1) = \mathrm{HF}_{\mathbb{W}}(i) = \sum_{j=1}^s \binom{m_j+1}{2}$  and  $\mathrm{HF}_{\mathbb{V}}(i) = 2i + 1 + \mathrm{HF}_{\mathbb{W}}(i-2)$ . Thus

$$\mathrm{HF}_{\Omega_{\mathbb{W}}/K}^1(i) = 4 \sum_{j=1}^s \binom{m_j+1}{2} - 2i - 1 - \mathrm{HF}_{\mathbb{W}}(i-2).$$

(iv) If  $0 \leq i \leq r_{\mathbb{W}}$ , we have  $\mathrm{HF}_{\mathbb{V}}(i) = 2i + 1 + \mathrm{HF}_{\mathbb{W}}(i-2)$  and

$$\mathrm{HF}_{\Omega_{R_{\mathbb{W}}/K}^1}(i) = \mathrm{HF}_{\mathbb{W}}(i) + 3 \mathrm{HF}_{\mathbb{W}}(i-1) - 2i - 1 - \mathrm{HF}_{\mathbb{W}}(i-2).$$

Altogether, we have proved the formula for the Hilbert function of  $\Omega_{R_{\mathbb{W}}/K}^1$  in the case  $\mu \geq 2\varrho + 4$ .

(b) Next we consider the case  $\mu \leq 2\varrho + 3$ . In this case, the relation between Hilbert functions of  $\mathbb{V}$  and of  $\mathbb{Y}$  follows from [Cat, Theorem 3.1]. Also, we have  $r_{\mathbb{V}} = \varrho + 1$  and  $r_{\mathbb{W}} = \varrho - 1 < r_{\mathbb{V}}$ , and hence  $\text{ri}(\Omega_{R_{\mathbb{W}}/K}^1) = r_{\mathbb{V}}$ . Therefore a similar argument as in the first case yields the desired formula for the Hilbert function of  $\Omega_{R_{\mathbb{W}}/K}^1$ .  $\square$

It is worth noting that [Cat, Theorem 3.1] and Theorem 9.2 give us a procedure for computing the Hilbert function of the module of Kähler differential 1-forms of  $R_{\mathbb{W}}/K$  from some suitable fat point schemes. Moreover,  $\text{HF}_{\Omega_{R_{\mathbb{W}}/K}^1}$  is completely determined by  $s$  and the multiplicities  $m_1, \dots, m_s$ .

**Remark 9.3.** On a non-singular conic  $\mathcal{C}$ , let  $\mathbb{W}$  be a complete intersection of type  $(2, d)$ . Let  $P \in \mathbb{W}$ , and let  $\mathbb{Y} = \mathbb{W} \setminus \{P\}$ . The regularity index of the scheme  $\mathbb{Y}$  is  $d - 1$ . Using Theorem 9.2 we see that the Hilbert function of  $\Omega_{R_{\mathbb{Y}}/K}^1$  is independent of the choice of the point  $P$ .

The following proposition can be used to find out a connection between the Hilbert functions of  $\Omega_{R_{\mathbb{W}}/K}^3$  (as well as of  $\Omega_{R_{\mathbb{W}}/K}^2$ ) and of a suitable subscheme of  $\mathbb{W}$  if  $\mathbb{W}$  is an equimultiple fat point scheme, i.e., if  $m_1 = \dots = m_s = \nu$ . This proposition follows from [Cat, Proposition 4.3].

**Proposition 9.4.** *In the setting of Theorem 9.2, we define*

$$\mathbb{Y} = \max\{m_1 - 1, 0\}P_1 + \dots + \max\{m_s - 1, 0\}P_s$$

*if  $\mu - s \geq 2\varrho$  and*

$$\mathbb{Y} = m_1P_1 + \dots + m_{s-2}P_{s-2} + \max\{m_{s-1} - 1, 0\}P_{s-1} + \max\{m_s - 1, 0\}P_s$$

*otherwise. Further, let  $q = \max\{\varrho, \lfloor \frac{\mu-s+1}{2} \rfloor\}$ , let  $\{G_1, \dots, G_r\}$  be a minimal homogeneous system of generators of  $I_{\mathbb{Y}}$ , and let  $L$  be the linear form passing through  $P_{s-1}$  and  $P_s$ .*

- (a) *If  $\mu - s \geq 2\varrho$  and  $\mu - s$  is odd, there exist  $F_1, F_2 \in (I_{\mathbb{W}})_q$  such that the set  $\{CG_1, \dots, CG_r, F_1, F_2\}$  is a minimal homogeneous system of generators of  $I_{\mathbb{W}}$ .*
- (b) *If  $\mu - s \geq 2\varrho$  and  $\mu - s$  is even, there exists  $F \in (I_{\mathbb{W}})_q$  such that the set  $\{CG_1, \dots, CG_r, F\}$  is a minimal homogeneous system of generators of  $I_{\mathbb{W}}$ .*
- (c) *If  $\mu - s < 2\varrho$ , there exists  $G \in (I_{\mathbb{W}})_q$  such that the set  $\{LG_1, \dots, LG_r, G\}$  is a minimal homogeneous system of generators of  $I_{\mathbb{W}}$ .*

In particular, if  $\mathbb{X} = \{P_1, \dots, P_s\}$  is a set of  $s$  distinct  $K$ -rational points on a non-singular conic  $\mathcal{C}$ , then, for every  $k \geq 1$ , there is a minimal homogeneous system of generators of  $I_{k\mathbb{X}}$  of the following form.

**Corollary 9.5.** *Let  $s \geq 4$ , let  $\mathbb{X} = \{P_1, \dots, P_s\}$  be a set of  $s$  distinct  $K$ -rational points on a non-singular conic  $\mathcal{C} = \mathcal{Z}^+(C)$ , let  $\{C, G_1, \dots, G_t\}$  be a minimal homogeneous system of generators of  $I_{\mathbb{X}}$ , and let  $k \geq 1$ .*

- (a) *If  $s = 2v$  for some  $v \in \mathbb{N}$ , then there exists a minimal homogeneous system of generators of  $I_{k\mathbb{X}}$  of the form*

$$\{C^k, C^{k-1}G_1, \dots, C^{k-1}G_t, C^{k-2}F_{21}, C^{k-3}F_{31}, \dots, CF_{(k-1)1}, F_{k1}\}$$

*where  $F_{j1} \in (I_{j\mathbb{X}})_{jv}$  for all  $j = 2, \dots, k$ .*

- (b) If  $s = 2v + 1$  for some  $v \in \mathbb{N}$  and  $k$  is even, then there exists a minimal homogeneous system of generators of  $I_{k\mathbb{X}}$  of the form

$$\{C^k, C^{k-1}G_1, \dots, C^{k-1}G_t, C^{k-2}F_{21}, \\ C^{k-3}F_{31}, C^{k-3}F_{32}, \dots, CF_{(k-1)1}, CF_{(k-1)2}, F_{k1}\}$$

where  $F_{jl} \in (I_{j\mathbb{X}})_{q_j}$  with  $q_j = \lfloor \frac{j(2v+1)+1}{2} \rfloor$  for every  $j = 2, \dots, k$  and  $l = 1, 2$ .

- (c) If  $s = 2v + 1$  for some  $v \in \mathbb{N}$  and  $k$  is odd, then there exists a minimal homogeneous system of generators of  $I_{k\mathbb{X}}$  of the form

$$\{C^k, C^{k-1}G_1, \dots, C^{k-1}G_t, C^{k-2}F_{21}, \\ C^{k-3}F_{31}, C^{k-3}F_{32}, \dots, CF_{(k-1)1}, F_{k1}, F_{k2}\}$$

where  $F_{jl} \in (I_{j\mathbb{X}})_{q_j}$  with  $q_j = \lfloor \frac{j(2v+1)+1}{2} \rfloor$  for every  $j = 2, \dots, k$  and  $l = 1, 2$ .

*Proof.* Since  $s \geq 4$ , we have  $q_k = \max\{2k, \lfloor \frac{sk+1}{2} \rfloor\} = \lfloor \frac{sk+1}{2} \rfloor$  for every  $k \geq 1$ . By Proposition 9.4 and by induction on  $k$ , we get the claimed minimal homogeneous system of generators of the ideal  $I_{k\mathbb{X}}$ .  $\square$

Now we present a relation between the Hilbert functions of the module of Kähler differential 3-forms  $\Omega_{R_{\nu\mathbb{X}}/K}^3$  and of  $S/\mathfrak{M}I_{(\nu-1)\mathbb{X}}$ , where  $\mathfrak{M} = \langle X_0, \dots, X_n \rangle$  is the homogeneous maximal ideal of  $S$ . Here we make the convention that  $I_{(\nu-1)\mathbb{X}} := \langle 1 \rangle$  if  $\nu = 1$ .

**Theorem 9.6.** *Let  $s \geq 4$  and  $\nu \geq 1$ , and let  $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^2$  be a set of  $s$  distinct  $K$ -rational points which lie on a non-singular conic  $\mathcal{C} = \mathcal{Z}^+(C)$ . Then we have  $\Omega_{R_{\nu\mathbb{X}}/K}^3 \cong (S/\mathfrak{M}I_{(\nu-1)\mathbb{X}})(-3)$ . In particular, for all  $i \in \mathbb{N}$ , we have*

$$\mathrm{HF}_{\Omega_{R_{\nu\mathbb{X}}/K}^3}(i) = \mathrm{HF}_{S/\mathfrak{M}I_{(\nu-1)\mathbb{X}}}(i-3).$$

*Proof.* Let  $\mathcal{B}_1 = \{C, G_1, \dots, G_t\}$  be a minimal homogeneous system of generators of  $I_{\mathbb{X}}$ , and let  $J^{(\nu)} = \langle \frac{\partial F}{\partial X_i} \mid F \in I_{\nu\mathbb{X}}, 0 \leq i \leq n \rangle$ . Note that  $r_{\mathbb{X}} = \lfloor \frac{s}{2} \rfloor$ . According to [GM, Proposition 1.1], we may assume that  $2 \leq \deg(G_j) \leq \lfloor \frac{s}{2} \rfloor + 1$  for  $j = 1, \dots, t$ . By Corollary 2.3, we have  $\Omega_{R_{\nu\mathbb{X}}/K}^3 \cong (S/J^{(\nu-1)})(-3)$ . Moreover, since  $\mathcal{C}$  is a non-singular conic, we have  $\langle \frac{\partial C}{\partial X_i} \mid 0 \leq i \leq 2 \rangle = \mathfrak{M}$ , and hence  $J^{(1)} = \mathfrak{M}$ . Thus it suffices to prove the equality  $J^{(\nu+1)} = \mathfrak{M}I_{\nu\mathbb{X}}$  for all  $\nu \geq 1$ . For  $k \geq 2$ , let  $\mathcal{B}_k$  be the minimal homogeneous system of generators of  $I_{k\mathbb{X}}$  constructed in Corollary 9.5. We see that  $\deg(F_{(k+1)1}) \geq \deg(F_{k1}) + 2$  for all  $k \geq 2$  and that  $\deg(F_{21}) = s$ .

If  $s = 4$  then  $\mathbb{X}$  is a complete intersection of type  $(2, 2)$ , and so we may assume  $\mathcal{B}_1 = \{C, G_1\}$  with  $\deg(G_1) = 2$ . This implies  $\deg(F_{21}) = \deg(G_1) + 2$ . In the case  $s \geq 5$  we have  $\deg(F_{21}) = s \geq \lfloor \frac{s}{2} \rfloor + 3 \geq \max\{\deg(G_j) \mid 1 \leq j \leq t\} + 2$ . Hence the inclusion  $J^{(\nu+1)} \subseteq \mathfrak{M}I_{\nu\mathbb{X}}$  holds true for all  $\nu \geq 1$ .

Now we proceed by induction on  $\nu$  to prove that  $C^{k-1}\mathfrak{M}I_{\nu\mathbb{X}} \subseteq J^{(\nu+k)}$  for all  $k \geq 1$ . In the following, let  $k \geq 1$ ,  $0 \leq i \leq 2$  and  $1 \leq j \leq t$ . If  $\nu = 1$ , we have

$$C^k\mathfrak{M} = C^k \langle \frac{\partial C}{\partial X_i} \mid 0 \leq i \leq 2 \rangle = \langle \frac{\partial C^{k+1}}{\partial X_i} \mid 0 \leq i \leq 2 \rangle \subseteq J^{(1+k)}.$$

Since  $\frac{\partial G_j}{\partial X_i} \in \mathfrak{M}$ , we also have

$$kC^{k-1}G_j \frac{\partial C}{\partial X_i} = \frac{\partial(C^k G_j)}{\partial X_i} - C^k \frac{\partial G_j}{\partial X_i} \in J^{(1+k)}.$$

Hence we get  $C^{k-1}\mathfrak{M}_{I_{\mathbb{X}}} \subseteq J^{(1+k)}$  for all  $k \geq 1$ , and the claim holds true for  $\nu = 1$ . Next we assume that  $\nu \geq 2$  and that  $C^{k-1}\mathfrak{M}_{I_{\mathbb{X}}} \subseteq J^{(l+k)}$  for  $1 \leq l \leq \nu - 1$  and all  $k \geq 1$ . We distinguish the following two cases.

**Case (a):** Suppose that  $s$  is even. Using Corollary 9.5(a) we write

$$\mathcal{B}_{\nu} = \{C^{\nu}, C^{\nu-1}G_1, \dots, C^{\nu-1}G_t, C^{\nu-2}F_{21}, \dots, CF_{\nu-1,1}, F_{\nu,1}\}$$

where  $F_{l1} \in (I_{\mathbb{X}})_{sl/2}$ . Note that  $\frac{\partial F_{l1}}{\partial X_i} \in \mathfrak{M}_{I_{(l-1)\mathbb{X}}}$  for  $2 \leq l \leq \nu$ . It follows from the inductive hypothesis that  $C^{k+\nu-l}\frac{\partial F_{l1}}{\partial X_i} \in C^{k+\nu-l}\mathfrak{M}_{I_{(l-1)\mathbb{X}}} \subseteq J^{(\nu+k)}$ . Thus, for  $2 \leq l \leq \nu$ , we have

$$(k + \nu - l)C^{k-1}(C^{\nu-l}F_{l1})\frac{\partial C}{\partial X_i} = \frac{\partial(C^{k+\nu-l}F_{l1})}{\partial X_i} - C^{k+\nu-l}\frac{\partial F_{l1}}{\partial X_i} \in J^{(\nu+k)}.$$

As above we have  $C^{k-1+\nu}\mathfrak{M} \subseteq J^{(\nu+k)}$  and  $C^{k+\nu-2}G_1\mathfrak{M} \subseteq J^{(\nu+k)}$ . Therefore we obtain  $C^{k-1}\mathfrak{M}_{I_{\nu\mathbb{X}}} \subseteq J^{(\nu+k)}$  for all  $k \geq 1$ , as desired.

**Case (b):** Suppose that  $s$  is odd. In this case the minimal homogeneous system of generators  $\mathcal{B}_{\nu}$  of  $I_{\nu\mathbb{X}}$  is given by

$$\begin{aligned} \mathcal{B}_{\nu} = \{ & C^{\nu}, C^{\nu-1}G_1, \dots, C^{\nu-1}G_t, C^{\nu-2}F_{21}, \\ & C^{\nu-3}F_{31}, C^{\nu-3}F_{32}, \dots, CF_{(2l-1)1}, CF_{(2l-1)2}, F_{2l1}\} \end{aligned}$$

if  $\nu = 2l$  and

$$\begin{aligned} \mathcal{B}_{\nu} = \{ & C^{\nu}, C^{\nu-1}G_1, \dots, C^{\nu-1}G_t, C^{\nu-2}F_{21}, \\ & C^{\nu-3}F_{31}, C^{\nu-3}F_{32}, \dots, CF_{2l1}, F_{(2l+1)1}, F_{(2l+1)2}\} \end{aligned}$$

if  $\nu = 2l + 1$ , where  $F_{ju} \in (I_{j\mathbb{X}})_{\lfloor \frac{js+1}{2} \rfloor}$  (see Corollary 9.5(b),(c)). Thus we can use the same argument as in case (a) and get  $C^{k-1}\mathfrak{M}_{I_{\nu\mathbb{X}}} \subseteq J^{(\nu+k)}$  for all  $k \geq 1$ .

Altogether, we have shown that  $C^{k-1}\mathfrak{M}_{I_{\nu\mathbb{X}}} \subseteq J^{(\nu+k)}$  for all  $\nu, k \geq 1$ . In particular, if  $k = 1$ , then we have  $\mathfrak{M}_{I_{\nu\mathbb{X}}} \subseteq J^{(\nu+1)}$ , and the proof is complete.  $\square$

If  $\nu = 1$ , we have  $\text{HF}_{\Omega_{R_{\mathbb{X}}/K}^3}(i) = 0$  for  $i \neq 3$  and  $\text{HF}_{\Omega_{R_{\mathbb{X}}/K}^3}(3) = 1$ . If  $\nu \geq 2$ , this Hilbert function can be described explicitly as follows.

**Proposition 9.7.** *In the setting of Theorem 9.6, let  $\mathcal{B}_1 = \{C, G_1, \dots, G_t\}$  be a minimal homogeneous system of generators of  $I_{\mathbb{X}}$  as constructed in Corollary 9.5, let  $d_j = \deg(G_j)$  for  $j = 1, \dots, t$ , and let  $\nu \geq 2$ . Suppose that  $d_1 \leq \dots \leq d_t$ . Then we have*

$$\text{HF}_{\Omega_{R_{\nu\mathbb{X}}/K}^3}(i) = \begin{cases} s\binom{\nu}{2} + h_i + \delta_i & \text{if } i \geq \lfloor \frac{s(\nu-1)}{2} \rfloor + 3, \\ \text{HF}_{\nu\mathbb{X}}(i-1) - 2i + 1 + h_i + \delta_i & \text{if } 2 < i < \lfloor \frac{s(\nu-1)}{2} \rfloor + 3, \\ 0 & \text{if } i \leq 2. \end{cases}$$

Here we let  $h_i = \#\{G \in \mathcal{B}_1 \mid \deg(G) = i + 1 - 2\nu\}$ , and  $\delta_i$  is defined as follows.

- (a) If  $s = 4$  then  $\delta_i = \nu - 2$  if  $i = 2\nu + 1$  and  $\delta_i = 0$  otherwise.
- (b) If  $s = 5$  then

$$\delta_i = \begin{cases} 1 & \text{if } i = 2\nu + 1, \\ 3 & \text{if } \nu \text{ is odd and } 2\nu + 3 \leq i \leq \frac{5\nu+1}{2}, \\ 3 & \text{if } \nu \text{ is even and } 2\nu + 3 \leq i < \frac{5\nu+2}{2}, \\ 2 & \text{if } \nu \text{ is even and } i = \frac{5\nu+2}{2}, \\ 0 & \text{otherwise.} \end{cases}$$



(c) If  $s \geq 6$  then

$$\delta_i = \begin{cases} 1 & \text{if } s \text{ is even and } i = 2\nu - 2k + \frac{ks}{2} + 1, 2 \leq k \leq \nu - 1, \\ 1 & \text{if } s \text{ is odd and } i = 2\nu + k(s-4) + 1, 1 \leq k \leq \lfloor \frac{\nu-1}{2} \rfloor, \\ 2 & \text{if } s \text{ is odd and } i = 2\nu + k(s-4) + \frac{s+1}{2} - 1, 1 \leq k \leq \lfloor \frac{\nu-2}{2} \rfloor, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* By Theorem 9.6, we have

$$\begin{aligned} \mathrm{HF}_{\Omega_{R_{\nu\mathbb{X}}/K}^3}(i) &= \mathrm{HF}_{S/\mathfrak{M}_{I_{(\nu-1)\mathbb{X}}}}(i-3) \\ &= \mathrm{HF}_S(i-3) - \mathrm{HF}_{\mathfrak{M}_{I_{(\nu-1)\mathbb{X}}}}(i-3) \\ &= \mathrm{HF}_S(i-3) - \dim_K(\mathfrak{M}_1(I_{(\nu-1)\mathbb{X}})_{i-4}) \\ &= \mathrm{HF}_S(i-3) - (\dim_K(I_{(\nu-1)\mathbb{X}})_{i-3} - \#(\mathcal{B}_{\nu-1})_{i-3}) \\ &= \mathrm{HF}_{(\nu-1)\mathbb{X}}(i-3) + \#(\mathcal{B}_{\nu-1})_{i-3}. \end{aligned}$$

for all  $i \in \mathbb{Z}$ . Note that we have  $r_{(\nu-1)\mathbb{X}} = \lfloor \frac{s(\nu-1)}{2} \rfloor$ . If  $i \geq r_{(\nu-1)\mathbb{X}} + 3$ , we obtain  $\mathrm{HF}_{(\nu-1)\mathbb{X}}(i-3) = s \binom{\nu}{2}$ . Otherwise, we have  $\mathrm{HF}_{(\nu-1)\mathbb{X}}(i-3) = \mathrm{HF}_{\nu\mathbb{X}}(i-1) - 2i + 1$  by [Cat, Theorem 3.1]. For every  $i \geq 0$ , we set

$$\delta_i := \#(\mathcal{B}_{\nu-1})_{i-3} - \#\{G \in \mathcal{B}_1 \mid \deg(G) = i + 1 - 2\nu\} = \#(\mathcal{B}_{\nu-1})_{i-3} - h_i.$$

Hence the claimed shape of the Hilbert function of  $\Omega_{R_{\nu\mathbb{X}}/K}^3$  follows immediately. Now we apply Corollary 9.5 to compute the values of  $\delta_i$ . We look at degrees of the elements in the tuple

$$\mathcal{A} = (C^{\nu-3}F_{21}, C^{\nu-4}F_{31}, \dots, CF_{(\nu-2)1}, F_{(\nu-1)1})$$

and get the tuple of degrees

$$\mathcal{A}^* = (2(\nu-3) + \lfloor \frac{2s+1}{2} \rfloor, 2(\nu-4) + \lfloor \frac{3s+1}{2} \rfloor, \dots, 2 + \lfloor \frac{(\nu-2)s+1}{2} \rfloor, \lfloor \frac{(\nu-1)s+1}{2} \rfloor).$$

Consider the following cases.

(a) Suppose that  $s = 4$ . Then  $\mathcal{B}_1 = \{C, G_1\}$  with  $\deg(G_1) = 2$  and every element in  $\mathcal{A}^*$  equals  $2\nu - 2$ . Also,  $\#(\mathcal{B}_{\nu-1})_{i-3} > 0$  only if  $i - 3 = 2\nu - 2$ . In the case  $i = 2\nu + 1$ , we have  $\#(\mathcal{B}_{\nu-1})_{i-3} = \nu$  and  $h_i = 2$ , and so  $\delta_i = \nu - 2$ . Clearly,  $\delta_i = 0$  if  $i \neq 2\nu + 1$ .

(b) Suppose that  $s = 5$ . We write  $\deg(C^{\nu-(k+1)}F_{k1}) = 2\nu - 2(k+1) + \lfloor \frac{5k+1}{2} \rfloor$  for  $k = 2, \dots, \nu - 1$ . It is easy to see that  $\deg(C^{\nu-(k+1)}F_{k1}) = \deg(C^{\nu-(k+2)}F_{k+11})$  if  $k$  is odd and  $\deg(C^{\nu-(k+1)}F_{k1}) + 1 = \deg(C^{\nu-(k+2)}F_{k+11})$  otherwise.

(i) If  $\nu$  is even then we have

$$\delta_i = \begin{cases} 1 & \text{if } i = 2\nu + 2, \\ 3 & \text{if } 2\nu + 3 \leq i < \frac{5\nu+2}{2}, \\ 2 & \text{if } i = \frac{5\nu+2}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If  $\nu$  is odd then we have

$$\delta_i = \begin{cases} 1 & \text{if } i = 2\nu + 2, \\ 3 & \text{if } 2\nu + 3 \leq i \leq \frac{5\nu+1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

(c) Suppose that  $s \geq 6$ . In this case the sequence of elements in  $\mathcal{A}^*$  is strictly increasing.

(i) If  $s$  is even then

$$\delta_i = \begin{cases} 1 & \text{if } i = 2\nu - 2k + \frac{ks}{2} + 1, 2 \leq k \leq \nu - 1, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If  $s$  is odd then

$$\delta_i = \begin{cases} 1 & \text{if } i = 2\nu + k(s-4) + 1, 1 \leq k \leq \lfloor \frac{\nu-1}{2} \rfloor, \\ 2 & \text{if } i = 2\nu + k(s-4) + \frac{s+1}{2} - 1, 1 \leq k \leq \lfloor \frac{\nu-2}{2} \rfloor, \\ 0 & \text{otherwise.} \end{cases}$$

Altogether, the claims follow.  $\square$

**Example 9.8.** Let  $K = \mathbb{Q}$ , and let  $\mathbb{X} = \{P_1, \dots, P_8\} \subseteq \mathbb{P}^2$  be the set of 8 points given by  $P_1 = (1 : 1 : 0)$ ,  $P_2 = (1 : 3 : 0)$ ,  $P_3 = (1 : 0 : 1)$ ,  $P_4 = (1 : 4 : 1)$ ,  $P_5 = (1 : 0 : 3)$ ,  $P_6 = (1 : 1 : 4)$ ,  $P_7 = (1 : 4 : 3)$ , and  $P_8 = (1 : 3 : 4)$ . Then  $\mathbb{X}$  is contained in the non-singular conic defined by  $C = 3X_0^2 - 4X_0X_1 + X_1^2 - 4X_0X_2 + X_2^2$ . In particular,  $\mathbb{X}$  is a complete intersection of type  $(2, 4)$ , and hence the set  $\mathcal{B}_1$  constructed in Corollary 9.5 is of the form  $\mathcal{B}_1 = \{C, G_1\}$  with  $\deg(G_1) = 4$ . The Hilbert functions of  $\nu\mathbb{X}$ ,  $1 \leq \nu \leq 3$ , are given by

$$\begin{aligned} \text{HF}_{\mathbb{X}} : & \quad 1 \ 3 \ 5 \ 7 \ 8 \ 8 \dots \\ \text{HF}_{2\mathbb{X}} : & \quad 1 \ 3 \ 6 \ 10 \ 14 \ 18 \ 21 \ 23 \ 24 \ 24 \dots \\ \text{HF}_{3\mathbb{X}} : & \quad 1 \ 3 \ 6 \ 10 \ 15 \ 21 \ 27 \ 33 \ 38 \ 42 \ 45 \ 47 \ 48 \ 48 \dots \end{aligned}$$

Moreover, we have  $\text{HF}_{\Omega_{R_{\mathbb{X}}/K}^3} : 0 \ 0 \ 0 \ 1 \ 0 \ 0 \dots$ . Now we apply Proposition 9.7 to compute the Hilbert functions of  $\Omega_{R_{\nu\mathbb{X}}/K}^3$ , where  $\nu = 2, 3$ . Since  $s = 8$  is even, we have

$$\delta_i = \begin{cases} 1 & \text{if } i = 2\nu + 2k + 1, 2 \leq k \leq \nu - 1, \\ 0 & \text{otherwise.} \end{cases}$$

If  $\nu = 2$ , then we have  $\delta_i = 0$  for all  $i \geq 1$  and

$$h_i = \begin{cases} 1 & \text{if } i = 5, 7, \\ 0 & \text{otherwise.} \end{cases}$$

An application of Proposition 9.7 yields

$$\text{HF}_{\Omega_{R_{2\mathbb{X}}/K}^3}(i) = \begin{cases} 8 + h_i + \delta_i & \text{if } i \geq 7, \\ \text{HF}_{2\mathbb{X}}(i-1) + h_i + \delta_i + 1 - 2i & \text{if } 2 < i < 7, \\ 0 & \text{if } i \leq 2. \end{cases}$$

Hence we get  $\text{HF}_{\Omega_{R_{2\mathbb{X}}/K}^3} : 0 \ 0 \ 0 \ 1 \ 3 \ 6 \ 7 \ 9 \ 8 \ 8 \dots$ .

Similarly, for  $\nu = 3$  we have

$$\delta_i = \begin{cases} 1 & \text{if } i = 11, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad h_i = \begin{cases} 1 & \text{if } i = 7, 9, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we get  $\text{HF}_{\Omega_{R_{3\mathbb{X}}/K}^3} : 0 \ 0 \ 0 \ 1 \ 3 \ 6 \ 10 \ 15 \ 18 \ 23 \ 25 \ 24 \ 24 \dots$ .

**Proposition 9.9.** *Let  $s \geq 4$  and  $\nu \geq 1$ , and let  $\mathbb{X} = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^2$  be a set of  $s$  distinct  $K$ -rational points which lie on a non-singular conic  $\mathcal{C} = \mathcal{Z}^+(C)$ . For  $i \geq 0$ , let  $h_i, \delta_i$  be defined as in Proposition 9.7.*

(a) *If  $\nu = 1$ , we have*

$$\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^2}(i) = \begin{cases} 0 & \text{if } i \geq s, \\ 3 \mathrm{HF}_{\mathbb{X}}(2) - 9 & \text{if } i = 3, \\ 3 \mathrm{HF}_{\mathbb{X}}(i-1) - \mathrm{HF}_{\mathbb{X}}(i-2) - 2i - 1 & \text{if } i < s \text{ and } i \neq 3. \end{cases}$$

(b) *For every  $\nu \geq 2$ , let  $\mu = \lfloor \frac{s\nu}{2} \rfloor + 2$  and  $t = \lfloor \frac{s(\nu-1)}{2} \rfloor + 3$ . Then we have*

$$\mathrm{HF}_{\Omega_{R_{\nu\mathbb{X}}/K}^2}(i) = \begin{cases} \frac{s(3\nu+2)(\nu-1)}{2} + h_i + \delta_i & \text{if } i \geq \lfloor \frac{s(\nu+1)}{2} \rfloor, \\ \frac{sv(3\nu+1)}{2} + h_i + \delta_i - 2i - 1 & \text{if } \mu \leq i < \lfloor \frac{s(\nu+1)}{2} \rfloor, \\ 3 \mathrm{HF}_{\nu\mathbb{X}}(i-1) - \mathrm{HF}_{\nu\mathbb{X}}(i-2) + s\binom{\nu}{2} + h_i + \delta_i - 2i - 1 & \text{if } t \leq i < \mu, \\ 4 \mathrm{HF}_{\nu\mathbb{X}}(i-1) - \mathrm{HF}_{\nu\mathbb{X}}(i-2) - 4i + h_i + \delta_i & \text{if } i < t. \end{cases}$$

*Proof.* Clearly,  $\mathrm{HF}_{\Omega_{R_{\nu\mathbb{X}}/K}^2}(i) = 0$  for  $i \leq 1$ . By Proposition 2.4, we have an exact sequence of graded  $R_{\nu\mathbb{X}}$ -modules

$$0 \longrightarrow \Omega_{R_{\nu\mathbb{X}}/K}^3 \longrightarrow \Omega_{R_{\nu\mathbb{X}}/K}^2 \longrightarrow \Omega_{R_{\nu\mathbb{X}}/K}^1 \longrightarrow \mathfrak{m}_{\nu\mathbb{X}} \longrightarrow 0.$$

For  $i \geq 2$ , we get

$$\begin{aligned} \mathrm{HF}_{\Omega_{R_{\nu\mathbb{X}}/K}^2}(i) &= \mathrm{HF}_{\Omega_{R_{\nu\mathbb{X}}/K}^1}(i) + \mathrm{HF}_{\Omega_{R_{\nu\mathbb{X}}/K}^3}(i) - \mathrm{HF}_{\nu\mathbb{X}}(i) \\ &\stackrel{(*)}{=} \mathrm{HF}_{\nu\mathbb{X}}(i) + 3 \mathrm{HF}_{\nu\mathbb{X}}(i-1) - \mathrm{HF}_{(\nu+1)\mathbb{X}}(i) + \mathrm{HF}_{\Omega_{R_{\nu\mathbb{X}}/K}^3}(i) - \mathrm{HF}_{\nu\mathbb{X}}(i) \\ &= 3 \mathrm{HF}_{\nu\mathbb{X}}(i-1) - \mathrm{HF}_{(\nu+1)\mathbb{X}}(i) + \mathrm{HF}_{\Omega_{R_{\nu\mathbb{X}}/K}^3}(i) \end{aligned}$$

where  $(*)$  follows from [KLL, Corollary 1.9(i)].

We consider the case  $\nu = 1$ . We see that  $\mathrm{HF}_{2\mathbb{X}}(3) = 10$ , so  $\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^2}(3) = 3 \mathrm{HF}_{\mathbb{X}}(2) - 10 + 1 = 3 \mathrm{HF}_{\mathbb{X}}(2) - 9$ . For  $i < s$  and  $i \neq 3$ , we have  $\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^2}(i) = 3 \mathrm{HF}_{\mathbb{X}}(i-1) - (2i+1 + \mathrm{HF}_{\mathbb{X}}(i-2))$ . For  $i \geq s$  we get  $\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^2}(i) = 3 \mathrm{HF}_{\mathbb{X}}(i-1) - \mathrm{HF}_{2\mathbb{X}}(i) = 3s - 3s = 0$ . Thus claim (a) follows.

Now we suppose  $\nu \geq 2$ . By Proposition 9.7 we have

$$\begin{aligned} \mathrm{HF}_{\Omega_{R_{\nu\mathbb{X}}/K}^2}(i) &= 3s \binom{\nu+1}{2} - s \binom{\nu+2}{2} + s \binom{\nu}{2} + h_i + \delta_i \\ &= \frac{s(3\nu+2)(\nu-1)}{2} + h_i + \delta_i \end{aligned}$$

for  $i \geq \lfloor \frac{s(\nu+1)}{2} \rfloor$ . If  $i < \lfloor \frac{s(\nu+1)}{2} \rfloor$ , then  $\mathrm{HF}_{(\nu+1)\mathbb{X}}(i) = 2i+1 + \mathrm{HF}_{\nu\mathbb{X}}(i-2)$  by [Cat, Theorem 3.1]. So, for  $\mu \leq i < \lfloor \frac{s(\nu+1)}{2} \rfloor$ , we have

$$\begin{aligned} \mathrm{HF}_{\Omega_{R_{\nu\mathbb{X}}/K}^2}(i) &= 2s \binom{\nu+1}{2} + s \binom{\nu}{2} + h_i + \delta_i - 2i - 1 \\ &= \frac{sv(3\nu+1)}{2} + h_i + \delta_i - 2i - 1. \end{aligned}$$

For  $t \leq i < \mu$ , we get

$$\mathrm{HF}_{\Omega_{R_{\nu\mathbb{X}}/K}^2}(i) = 3 \mathrm{HF}_{\nu\mathbb{X}}(i-1) - \mathrm{HF}_{\nu\mathbb{X}}(i-2) - 2i - 1 + s \binom{\nu}{2} + h_i + \delta_i.$$

For  $i < t$ , it follows from Proposition 9.7 again that

$$\begin{aligned} \mathrm{HF}_{\Omega_{R_{\nu\mathbb{X}}/K}^2}^2(i) &= 3 \mathrm{HF}_{\nu\mathbb{X}}(i-1) - (2i+1 + \mathrm{HF}_{\nu\mathbb{X}}(i-2)) \\ &\quad + (\mathrm{HF}_{\nu\mathbb{X}}(i-1) - 2i+1 + h_i + \delta_i) \\ &= 4 \mathrm{HF}_{\nu\mathbb{X}}(i-1) - \mathrm{HF}_{\nu\mathbb{X}}(i-2) - 4i + h_i + \delta_i. \end{aligned}$$

Therefore claim (b) is completely proved.  $\square$

To end this section, we apply the preceding proposition to compute the Hilbert function of  $\Omega_{R_{\nu\mathbb{X}}/K}^2$  in a concrete case.

**Example 9.10.** Let  $\mathbb{X}$  be the complete intersection given in Example 9.8. For  $1 \leq \nu \leq 3$ , an application of Proposition 9.9 yields

$$\begin{aligned} \mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^2}^2 &: 0 \ 0 \ 3 \ 6 \ 7 \ 6 \ 3 \ 1 \ 0 \ 0 \cdots \\ \mathrm{HF}_{\Omega_{R_{2\mathbb{X}}/K}^2}^2 &: 0 \ 0 \ 3 \ 9 \ 18 \ 27 \ 34 \ 39 \ 39 \ 38 \ 35 \ 33 \ 32 \ 32 \cdots \\ \mathrm{HF}_{\Omega_{R_{3\mathbb{X}}/K}^2}^2 &: 0 \ 0 \ 3 \ 9 \ 18 \ 30 \ 45 \ 60 \ 73 \ 84 \ 90 \ 95 \ 95 \ 94 \ 91 \ 89 \ 88 \ 88 \cdots \end{aligned}$$

## 10. THE KÄHLER DIFFERENTIAL ALGEBRA OF $R_{\mathbb{X}}/K[x_0]$

Given a 0-dimensional scheme  $\mathbb{X} \subseteq \mathbb{P}^n$  such that no point in its support is contained in the hyperplane  $\mathcal{Z}^+(X_0)$ , the element  $x_0 = X_0 + I_{\mathbb{X}}$  is a non-zerodivisor of  $R_{\mathbb{X}}$ . Hence there is a short exact sequence of graded  $R_{\mathbb{X}}$ -modules

$$0 \longrightarrow R_{\mathbb{X}} dx_0 \longrightarrow \Omega_{R_{\mathbb{X}}/K}^1 \longrightarrow \Omega_{R_{\mathbb{X}}/K[x_0]}^1 \longrightarrow 0$$

(see [Kun, Proposition 3.24]). As noted in Section 2, the homogeneous  $R_{\mathbb{X}}$ -linear map  $\gamma : \Omega_{R_{\mathbb{X}}/K}^1 \rightarrow R_{\mathbb{X}}$  satisfies  $\gamma(dx_i) = x_i$  for all  $i = 0, \dots, n$ . If  $f dx_0 = 0$  for some homogeneous element  $f \in R_{\mathbb{X}}$ , then  $\gamma(f dx_0) = f x_0 = 0$ , and so  $f = 0$ , since  $x_0$  is a non-zerodivisor of  $R_{\mathbb{X}}$ . Hence we have  $\mathrm{Ann}_{R_{\mathbb{X}}}(dx_0) = \langle 0 \rangle$ . Consequently, we obtain

$$\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K[x_0]}^1}^1(i) = \mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K}^1}^1(i) - \mathrm{HF}_{\mathbb{X}}(i-1)$$

for all  $i \in \mathbb{Z}$  and  $\mathrm{ri}(\Omega_{R_{\mathbb{X}}/K[x_0]}^1) \leq \max\{r_{\mathbb{X}} + 1, \mathrm{ri}(\Omega_{R_{\mathbb{X}}/K}^1)\}$ .

Thus the Hilbert functions of  $\Omega_{R_{\mathbb{X}}/K}^1$  and  $\Omega_{R_{\mathbb{X}}/K[x_0]}^1$  are strongly related, and it is straightforward to transfer our earlier results to results about  $\Omega_{R_{\mathbb{X}}/K[x_0]}^m$ . For instance, for a 0-dimensional subscheme of  $\mathbb{P}^1$ , an application of Proposition 3.1 yields the following property.

**Proposition 10.1.** *Let  $\mathbb{X} \subseteq \mathbb{P}^1$  be a 0-dimensional scheme, and let  $I_{\mathbb{X}} = \langle F \rangle$ , where  $F = \prod_{i=1}^s (X_1 - a_i X_0)^{m_i}$  for some  $s$ ,  $m_1, \dots, m_s \geq 1$ , and  $a_i \in K$  with  $a_i \neq a_j$  for  $i \neq j$ , and let  $\mu = \sum_{i=1}^s m_i$ . Then the Hilbert functions of the modules of Kähler differential  $m$ -forms of  $R_{\mathbb{X}}/K[x_0]$  are given by*

$$\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K[x_0]}^1}^1 : 0 \ 1 \ 2 \ 3 \ \cdots \ \mu-2 \ \mu-1 \ \mu-2 \ \mu-3 \ \cdots \ \mu-s \ \mu-s \cdots$$

and  $\mathrm{HF}_{\Omega_{R_{\mathbb{X}}/K[x_0]}^2}^2(i) = 0$  for all  $i \in \mathbb{Z}$ .

In general, for  $1 \leq m \leq n+1$ , the module of Kähler differential  $m$ -forms  $\Omega_{R_{\mathbb{X}}/K[x_0]}^m$  has a presentation  $\Omega_{R_{\mathbb{X}}/K[x_0]}^m \cong \Omega_{S/K[x_0]}^m / (I_{\mathbb{X}} \Omega_{S/K[x_0]}^m + d_{S/K[x_0]} I_{\mathbb{X}} \Omega_{S/K[x_0]}^{m-1})$  (cf. [Kun,

Proposition 4.12]). Moreover, according to [SS, X.83], the above exact sequence induces an exact sequence of graded  $R_{\mathbb{X}}$ -modules

$$0 \longrightarrow R_{\mathbb{X}} dx_0 \wedge_{R_{\mathbb{X}}} \Omega_{R_{\mathbb{X}}/K}^{m-1} \longrightarrow \Omega_{R_{\mathbb{X}}/K}^m \longrightarrow \Omega_{R_{\mathbb{X}}/K[x_0]}^m \longrightarrow 0.$$

By using these results and by applying the methods given in Propositions 4.2, 5.3, 6.2 and 7.4, we get the following properties for the Hilbert function and regularity index of  $\Omega_{R_{\mathbb{X}}/K[x_0]}^m$ . We leave the detailed proofs of these properties to the interested reader.

**Proposition 10.2.** *Let  $\mathbb{X} \subseteq \mathbb{P}_K^n$  be a 0-dimensional scheme, and let  $1 \leq m \leq n+1$ .*

- (a) *For  $i < m$ , we have  $\text{HF}_{\Omega_{R_{\mathbb{X}}/K[x_0]}^m}(i) = 0$ .*
- (b) *For  $m \leq i < \alpha_{\mathbb{X}} + m - 1$ , we have  $\text{HF}_{\Omega_{R_{\mathbb{X}}/K[x_0]}^m}(i) = \binom{n}{m} \cdot \binom{n+i-m}{n}$ .*
- (c) *The Hilbert polynomial of  $\Omega_{R_{\mathbb{X}}/K[x_0]}^m$  is constant.*
- (d) *We have  $\text{HF}_{\Omega_{R_{\mathbb{X}}/K[x_0]}^m}(r_{\mathbb{X}} + m) \geq \text{HF}_{\Omega_{R_{\mathbb{X}}/K[x_0]}^m}(r_{\mathbb{X}} + m + 1) \geq \dots$ , and if  $\text{ri}(\Omega_{R_{\mathbb{X}}/K[x_0]}^m) \geq r_{\mathbb{X}} + m$  then*

$$\text{HF}_{\Omega_{R_{\mathbb{X}}/K[x_0]}^m}(r_{\mathbb{X}} + m) > \dots > \text{HF}_{\Omega_{R_{\mathbb{X}}/K[x_0]}^m}(\text{ri}(\Omega_{R_{\mathbb{X}}/K[x_0]}^m)).$$

- (e) *The regularity indices of  $\Omega_{R_{\mathbb{X}}/K[x_0]}^m$  satisfies*

$$\text{ri}(\Omega_{R_{\mathbb{X}}/K[x_0]}^m) \leq \max\{r_{\mathbb{X}} + m, \text{ri}(\Omega_{R_{\mathbb{X}}/K}^1) + m - 1\}.$$

Finally, for non-reduced fat point schemes in  $\mathbb{P}^n$ , the Hilbert function and regularity index of  $\Omega_{R_{\mathbb{W}}/K[x_0]}^m$  have the following properties.

**Proposition 10.3.** *Let  $\mathbb{W} = m_1 P_1 + \dots + m_s P_s$  be a fat point scheme in  $\mathbb{P}^n$ , and let  $\mathbb{V}$  be the fattening of  $\mathbb{W}$ . Suppose that  $m_i \geq 2$  for some  $i \in \{1, \dots, s\}$ , and let  $1 \leq m \leq n+1$ .*

- (a) *The Hilbert polynomial of  $\Omega_{R_{\mathbb{W}}/K[x_0]}^m$  is bounded by*

$$\sum_{i=1}^s \binom{n}{m} \binom{m_i+n-2}{n} \leq \text{HP}_{\Omega_{R_{\mathbb{W}}/K[x_0]}^m}(z) \leq \sum_{i=1}^s \binom{n}{m} \binom{m_i+n-1}{n}.$$

- (b) *We have*

$$\text{ri}(\Omega_{R_{\mathbb{W}}/K[x_0]}^m) \leq \min\{\max\{r_{\mathbb{W}} + m, r_{\mathbb{V}} + m - 1\}, \max\{r_{\mathbb{W}} + n, r_{\mathbb{V}} + n - 1\}\}.$$

- (c) *If  $m_1 \leq \dots \leq m_s$  and  $\text{Supp}(\mathbb{W}) = \{P_1, \dots, P_s\}$  is in general position, then*

$$\text{ri}(\Omega_{R_{\mathbb{W}}/K[x_0]}^m) \leq \min \left\{ \max\{m_s + m_{s-1} + m, \lfloor \frac{\sum_{j=1}^s m_j + s + n - 2}{n} \rfloor + m - 1\}, \right. \\ \left. \max\{m_s + m_{s-1} + n, \lfloor \frac{\sum_{j=1}^s m_j + s + n - 2}{n} \rfloor + n - 1\} \right\}.$$

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